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Holographic correlators in a flow to a fixed point

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Abstract: Using holographic renormalization, we study correlation functions throughout a renormalization group flow between two-dimensional superconformal field theories. The ultraviolet theory is an $N = (4,4)$ CFT which can be thought of as a symmetric product of U(2) super WZW models. It is perturbed by a relevant operator which preserves one-quarter supersymmetry and drives the theory to an infrared fixed point. We compute correlators of the stress-energy tensor and of the relevant operators dual to supergravity scalars. Using the former, we put together Zamolodchikov's C function, and contrast it with proposals for a holographic C function. In passing, we address and resolve two puzzles also found in the case of five-dimensional bulk supergravity.

Keywords: Renormalization Group, AdS-CFT and dS-CFT Correspondence, Supergravity Models.

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Contents

1. Introduction

In the AdS/CFT correspondence, renormalization group (RG) flows of a d-dimensional conformal field theory are described by domain wall solutions of the dual $(d+1)$-dimensional bulk supergravity theory, see e.g. [1] and references therein. Physically, the domain wall solution can be thought of as a shell of matter, where the metric becomes asymptotically AdS far from the shell. In terms of limiting behavior of the dual field theory, the asymptotic AdS length scale gives the central charge at the ultraviolet conformal fixed point of the field theory. Should the field theory be conformal also in the infrared, the bulk space is asymptotically AdS also in the deep interior of the shell, with the inner AdS length related to the infrared central charge of the field theory. In the supergravity scalar target space this means that the flow does not run off to infinity but goes down to a minimum of the potential and stops. The solution is nonsingular everywhere, so for weak fluctuations and small curvature, the supergravity approximation may be trusted throughout the flow.

The previously studied five-dimensional examples of such conformal-to-conformal flows could only be given numerically [2]. This has remained an unsurmountable obstacle for further application of holographic renormalization methods, in particular for the computation of correlators in nonsingular flows. The few flows that *are* known exactly, on the other hand, all run off to infinity in the scalar target space, which creates singularities at finite distance in the bulk spacetime [3, 4]. On the field theory side, these solutions describe theories that confine in the infrared. In [5], for three-dimensional bulk, we found the presently only known exact solution describing an RG flow to an infrared fixed point. The present paper is about correlators in this smooth RG flow between conformal fixed points. Our main result is the computation of two-point functions of the stress-energy tensor and operators dual to the supergravity scalars.[1]

To be specific, the flow we study is a solution of the three-dimensional $\mathrm{SO}(4)\times\mathrm{SO}(4)$ gauged supergravity [6] with equal coupling constants and 16 supercharges. The dual field theory is a large (or "double") $N=(4,4)$ superconformal theory with $\mathrm{SU}(2)^4$ current algebra and equal levels of the two $\mathrm{SU}(2)^2$ factors. It has an alternative realization as a symmetric product of $\mathrm{U}(2)$, $N=1$ super WZW models [7], and in terms of branes, it is the worldvolume theory on the intersection (along the D1-branes) of two D1-D5 systems with equal D5-brane charge. This brane setup has near-horizon limit $AdS_3\times S^3\times S^3\times S^1$, where one can recognize the two $\mathrm{SO}(4)$'s as transverse rotations of the two brane systems. Finally, the RG flow we describe is driven by a relevant operator $\mathcal{O}_q$ of ultraviolet dimension $3/2$, that breaks conformal symmetry and leaves $1/4$ of the supersymmetry.

Let us briefly expand on our motivations for this work. On a purely methodological level, we want to explore the extent to which three-dimensional gauged supergravity can be used to describe deformations of interesting but complicated two-dimensional CFTs, such as worldvolume theories on intersecting D-branes. In the end, we hope to have provided an example that three-dimensional gauged supergravity does provide a powerful addition to the arsenal of methods in two-dimensional CFT. In particular, on the supergravity side we are able to compute quantities that are very difficult to compute in the deformed CFT,

[1] Guidelines for selected reading of this paper are given at the end of the introduction.

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e.g. $\langle TT \rangle$ correlators and the C function. It is intriguing that on the field theory side, those computations seem to be difficult but not impossible, which makes our results predictions for supersymmetric (and perhaps integrable) deformations of two-dimensional CFT. In addition to this motivation, our flow may be viewed as a toy model in which to address some questions raised in flows of higher dimensional theories, such as four-dimensional $N = 1$ super Yang-Mills theory. The toy model idea proves to be useful, since we do find some new insight into old mysteries encountered in holography with five-dimensional bulk supergravity (see section 5).

For completeness, let us mention that in [5] we did not just find the supersymmetric flow but we also found a stable nonsupersymmetric fixed point of this theory (in fact, the first stable nonsupersymmetric fixed point; later on several others were found in the three-dimensional maximal theory [8, 9]). A flow to such a nonsupersymmetric fixed point may be relevant to some aspects of black-hole physics, and this was part of the motivation of [5]. However, although we hope to discuss black-hole physics in future work, here we focus entirely on the supersymmetric flow and the complementary motivations mentioned in the previous paragraph.

To obtain the correlators, we use the formalism of holographic renormalization [10]–[13]. It is a framework to reliably compute correlators along RG flows of quantum field theories using the AdS/CFT correspondence. In particular, it allows one to compute one-point functions in the presence of sources, to check that they obey the requisite Ward identities throughout the flow, and to compute power-law terms which were usually dropped in the old prescription [14, 15]. In a theory that is conformal also in the infrared, such as the one considered here, surely a basic requirement of the formalism one wishes to apply is that it is sufficiently restrictive to single out the correct asymptotic power-law behavior automatically. This is true for holographic renormalization, as we show for example in section 5.4. As a consequence of being able to compute one-point functions, one can also make sure one is using a renormalization scheme that preserves supersymmetry, since $\langle T_{ij} \rangle = 0$ is then expected to hold in the background. Finite counterterms have to be added to ensure this, corresponding to a selection of a renormalization scheme that preserves supersymmetry in the dual field theory.

After this brief introduction to the formalism, we proceed to summarize the new results in this paper. We extend the analysis of [11, 12] to also include supermultiplets consisting of "inert" scalars Φ (meaning they have vanishing background — as opposed to the "active" scalar Q which carries the domain wall background). As it turns out, the treatment of the inert scalars raises some new conceptual questions, such as the proper choice of finite counterterms and the distinction between inert scalars with conformal dimensions Δ_+ and Δ_-, corresponding to the same mass in supergravity, as we shall shortly discuss. Through the coupling to the active scalar Q, the inert scalars change mass along the flow to the fixed point. This coupling also allows for (in fact, requires) a new type of finite counterterm, of the form $Q^2\Phi^2$. We compute the coefficients of these terms for all inert scalars, as well as the coefficient of the finite counterterm Q^4 (first displayed in [11]). The terms of type $Q^2\Phi^2$ can be seen as natural generalizations of the finite Q^4 term, but the coefficients of the $Q^2\Phi^2$ terms cannot be determined by evaluation on the background, since Φ itself vanishes on the background. In section 5.3 we will see how to compute those coefficients, using a supersymmetry Ward identity for the two-point functions of superpartner inert scalars; the result is listed in table 2. By adding all counterterms and taking the cutoff ϵ back to zero, we compute the renormalized action. As usual, all correlators are computed by functional differentiation of this renormalized action with respect to the boundary sources, and then setting the sources to zero.

The next result is more conceptual. It was noticed in [5] that the effective potential $\mathcal{V}$ appearing in the fluctuation equations of the inert scalars can be expressed in terms of a simple prepotential in the sense of supersymmetric quantum mechanics (susy-QM).[2] This property is very useful, since it guarantees the absence of tachyonic fluctuations. Similarly, such prepotentials were found in the known five-dimensional examples [16]; the general existence of susy-QM prepotentials seemed in need of further explanation. In section 5.1 we show that the existence of susy-QM prepotentials follows from the preserved $N = 1$ supersymmetry of the background flow, and that the prepotentials may be directly extracted from the fermionic mass term of the underlying gauged supergravity. This argument extends readily to higher dimensions. This is one aspect in which we see that the toy model does provide useful information in higher dimensions.

We then proceed to determine the supersymmetry Ward identity relating two-point functions of a pair of superpartner scalars. This is useful as it provides a way to distinguish between different conformal dimensions associated to the same supergravity mass. It is well known that scalars of mass in a certain range (here $-1 < m^2 \leq 0$) can correspond to two different solutions $\Delta_\pm$ for the conformal dimension of the dual operator. For the active scalar, the choice of Δ_+ or Δ_- is the difference between whether the background describes an operator flow or a vev flow [17], but this is not so for the inert scalars. Our scalars are precisely in this range, and unlike in previously studied cases, there are now two representation sectors with the same quantum numbers, so group theory is not sufficient to make the distinction. Using the fact that the correlator asymptotics depends directly on Δ and not just on the supergravity mass, and following the correlators from one fixed point to the other, we are able to distinguish between Δ_+ and Δ_-.

We then derive the fluctuation equations for the inert scalars, active scalar and metric around the domain wall solution. In all previously studied cases, those fluctuation equations were hypergeometric. Instead, we find that around our domain wall solution all fluctuation equations, for inert and active scalars as well as for the metric and vector fields, reduce to a slightly more complicated equation, the *biconfluent Heun equation*, which descends by confluence from a Fuchsian equation with four regular singularities. We devise some methods to solve this equation; the mathematics is relegated to appendix B where in particular we point out a simple and efficient way to compute the sought-after coefficient numerically. This allows us to achieve our main goal: the computation of two-point correlation functions throughout the renormalization group flow.

[2] In this paper, the notion of *prepotential* always refers to this supersymmetric quantum mechanics function $\mathcal{U}$ encoding the potential of a fluctuation equation as $\mathcal{V} = \mathcal{U}' + \mathcal{U}^2$, as opposed to the *superpotential* W which describes the background potential V of the active scalar as $V = \frac{1}{2}(W')^2 - 2W^2$.

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The final part of the paper concerns the computation of a C function, i.e. a function that is monotonic as a function of RG scale along the flow and interpolates between the central charges at the conformal fixed points. The general existence of such a function can be very useful to map out the space of field theories, for instance to find well-defined universality classes. In two dimensions, Zamolodchikov has given a general construction in terms of the two-point correlators of the stress-energy tensor [18]. Since Zamolodchikov's proof of monotonicity relies heavily on the lack of distinct tensor structures for stress-energy 2-point functions in two dimensions, it has no straightforward generalization to higher dimensions, but there have been several proposals for defining monotonic C functions by holography. In particular, in [19, 2] positive-energy conditions in the bulk were used to produce a monotonic boundary function in terms of the superpotential of the flow. It is now interesting that in our two-dimensional example, we have both these objects at our disposal: Zamolodchikov's C function in terms of the holographic correlators, as well as the holographic proposal of [19, 2].

The paper is organized as follows. In section 2 we review the exact domain wall solution of [5] that describes the flow. In section 3, we display the field equations and solve them perturbatively, i.e. close to the AdS boundary. In section 4, this solution is used to compute counterterms to form the renormalized action, which is then functionally differentiated to give the one-point functions. Solving the bulk fluctuation equations, this is already sufficient information to determine the two-point functions for inert scalars in section 5. In this section we also address the conceptual issues of prepotentials and distinction between Δ_+ and Δ_-. For the sector of active scalar and stress-energy tensor fluctuations, one needs to do some more work, since their mutual coupling requires one to find gauge invariant quantities to work with. This is done in section 6 and we compute their linearized fluctuation equations around the domain wall. Then, in section 7, we use the stress-energy correlators to study the Zamolodchikov C function and contrast it with other proposals for a C function.

Since this paper is relatively long, let us give some guidelines on how the reader can get the most out of it in the shortest amount of time, depending on his or her preferences.

- *Learn the formalism of holographic renormalization*: this paper provides an example that is technically less demanding than in the defining papers [10, 11, 12], where emphasis was on the 5d/4d case, yet is still quite different from the basic example of a free massive scalar in the review [13]. The reader with this interest would be well-advised to concentrate on sections 3 and 4; the philosophy is explained in the latter, with the core explanation in section 4.1. Then it is straightforward to derive inert correlators as in 5.4; the active/metric sector also requires decoupling their fluctuation equations, as in section 6.

- See *new conceptual issues that are also relevant in other dimensions*, like the inclusion of inert scalars, the discussion of prepotentials, and the distinction between Δ_+ and Δ_- using correlators. These issues are dealt with in section 5. The notation is fairly standard in the literature so skipping previous details should not encumber the expert reader much.

- Readers interested mainly in *deformations of CFT, not supergravity details* can enjoy our main results, the deformed 2-point functions, in section 5.4 and at the end of section 6.2. The notation is mostly self-explanatory, but it might be wise to simultaneously consult appendix B where $\Psi_\alpha(p)$ and the relevant special functions are explained. The C function is presented in section 7.

2. An exact holographic conformal-to-conformal flow

In this section, we briefly review the analytic domain wall solution of [5] which interpolates between two AdS vacua. It was constructed as a solution in the three-dimensional $N = 8$ gauged supergravity with local $\mathrm{SO}(4)\times\mathrm{SO}(4)$ symmetry [6], describing the $AdS_3\times S^3\times S^3\times S^1$ near-horizon geometry of the double D1-D5 system [20, 7] with equal D5-brane charges. The matter sector of this theory consists of n multiplets each containing 8 scalars and 8 fermions, whereas graviton, gravitini, and the 12 vector fields are non-propagating in three dimensions. The $8n$ scalars parametrize the coset manifold $\mathrm{SO}(8, n)/(\mathrm{SO}(8) \times \mathrm{SO}(n))$.

The supergravity lagrangian is given by [6]

$$\mathcal{L} = \frac{1}{4}\sqrt{G}R + \mathcal{L}_{\mathrm{CS}} + \frac{1}{4}\sqrt{G}G^{\mu\nu}\,\mathcal{P}_\mu^{Ir}\mathcal{P}_\nu^{Ir} + \sqrt{G}\,V + \mathcal{L}_{\mathrm{F}}\,, \tag{2.1}$$

where $\mathcal{L}_{\mathrm{CS}}$ is the Chern-Simons term for the vector fields, the third term is the kinetic term for the scalars, given explicitly in eq. (3.3) below, V denotes the scalar potential and finally $\mathcal{L}_F$ contains the fermionic terms, given in [5]. We use indices $I, J, \ldots$ and indices $r, s, \ldots$ to label the vector representations of SO(8) and SO(n), respectively. The 12 vector fields transform in the adjoint representation of the gauge group

$$\mathrm{SO}(4)^+ \times \mathrm{SO}(4)^- \;\subset\; \mathrm{SO}(8) \;\subset\; \mathrm{SO}(8)\times\mathrm{SO}(n) \;\subset\; \mathrm{SO}(8,n)\,, \tag{2.2}$$

where we use superscripts $\pm$ to distinguish the two three-spheres. In addition to the local gauge symmetry, the theory is invariant under the rigid action of SO(n). Assuming $n \geq 4$ matter multiplets, we break the latter down to $\mathrm{SO}(4)\times\mathrm{SO}(n-4)$ and consider the following subgroup

$$\begin{aligned} G_{\mathrm{inv}} &\equiv \mathrm{SO}(4)_{\mathrm{inv}} \times \mathrm{SO}(n-4) \\ &\subset \big(\mathrm{SO}(4)^+ \times \mathrm{SO}(4)^-\big) \times \big(\mathrm{SO}(4) \times \mathrm{SO}(n-4)\big)\,, \end{aligned} \tag{2.3}$$

of the global invariance group of the potential V. The $\mathrm{SO}(4)_{\mathrm{inv}}$ factor in G_{inv} is embedded as the diagonal of the three SO(4) factors on the right hand side. Evaluation of the scalar potential V on the two-dimensional space of singlets under G_{inv} leads to the potential [5]

$$V = -g^2\left(16 + 24\,(Z_1^2 + Z_2^2) + 8\,(Z_1^2 + Z_2^2)^2 - 8\,(Z_1^6 + Z_2^6) - 4\,(Z_1^4 + Z_2^4)^2\right). \tag{2.4}$$

where g is the remaining gauge coupling constant.

The form of this potential is depicted in figure 1. It exhibits two inequivalent extremal points apart from the local maximum at the origin. The saddle point at $(Z_1, Z_2) = (1, 0)$ corresponds to a nonsupersymmetric but stable vacuum as has been verified by explicit

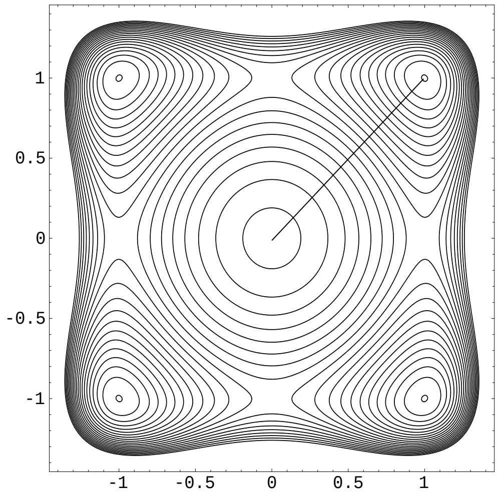

Figure 1: Contour plot of the scalar potential $V(Z_1, Z_2)$ (2.4) and the flow trajectory.

$\mathrm{SO}(4)_{\mathrm{inv}}$	$N = (1,1)$ multiplets $h \times \bar{h}$	field content $(h, \bar{h})$
(1,1)	$(\frac{5}{4}\vert\frac{7}{4}) \times (\frac{5}{4}\vert\frac{7}{4})$	$(\frac{5}{4},\frac{5}{4})$, $(\frac{7}{4},\frac{7}{4})$, $(\frac{5}{4},\frac{7}{4})$, $(\frac{7}{4},\frac{5}{4})$
(3,3)	$(\frac{1}{4}\vert\frac{3}{4}) \times (\frac{1}{4}\vert\frac{3}{4})$	$(\frac{1}{4},\frac{1}{4})$, $(\frac{3}{4},\frac{3}{4})$, $(\frac{1}{4},\frac{3}{4})$, $(\frac{3}{4},\frac{1}{4})$
(1,3)	$(\frac{1}{4}\vert\frac{3}{4}) \times (\frac{5}{4}\vert\frac{7}{4})$	$(\frac{1}{4},\frac{7}{4})$, $(\frac{1}{4},\frac{5}{4})$, $(\frac{3}{4},\frac{7}{4})$, $(\frac{3}{4},\frac{5}{4})$
(3,1)	$(\frac{5}{4}\vert\frac{7}{4}) \times (\frac{1}{4}\vert\frac{3}{4})$	$(\frac{7}{4},\frac{1}{4})$, $(\frac{5}{4},\frac{1}{4})$, $(\frac{7}{4},\frac{3}{4})$, $(\frac{5}{4},\frac{3}{4})$
(2,2)	$(\frac{1}{2}\vert 1) \times (\frac{1}{2}\vert 1)$	$(\frac{1}{2},\frac{1}{2})$, $(1,1)$, $(\frac{1}{2},1)$, $(1,\frac{1}{2})$

Table 1: Supergravity spectrum around the supersymmetric vacuum $(Z_1, Z_2) = (1, 1)$.

computation of the scalar fluctuations around this point [5]. We will concentrate on the extremum located at $(Z_1, Z_2) = (1, 1)$, which preserves $N = (1, 1)$ supersymmetry.

The ratio of the central charges of the dual conformal field theory at this extremum and that of the CFT at the origin is given by [21, 22]

$$\frac{c_{\mathrm{IR}}}{c_{\mathrm{UV}}} = \sqrt{\frac{V_{\mathrm{UV}}}{V_{\mathrm{IR}}}} = \frac{1}{2}\,, \tag{2.5}$$

supporting the conjecture that this point corresponds to a mass deformation of the UV conformal field theory; half the fields are integrated out to form the IR theory. The supergravity spectrum around this point is organized in $N = (1, 1)$ supermultiplets as summarized in table 1, where h and $\bar{h}$ denote the conformal dimensions associated with the supergravity masses, so that $\Delta = h + \bar{h}$. Note that the multiplet in the $(\mathbf{1}, \mathbf{1})$ for instance contains two scalars and spin-1/2 fields, whereas the multiplet in the $(\mathbf{1}, \mathbf{3})$ combines a spin-1/2 field with massive selfdual vectors and a massive gravitino.

An analytic domain wall solution interpolating between the origin and the supersymmetric extremum was constructed in [5] using the ansatz

$$Z_1 = Z_2 = \frac{1}{\sqrt{2}} \sinh\left(\frac{1}{\sqrt{2}} Q\right), \tag{2.6}$$

where Q denotes the active scalar field parametrizing the diagonal in figure 1. The remaining scalar fields are collectively referred to as *inert* scalars. Around the origin $Z_1 = Z_2 = 0$, the mass of the active scalar field is $m^2 L_0^2 = -3/4$, i.e. it is dual to a relevant operator of conformal dimension $\Delta = 3/2$ which drives the flow away from the UV conformal field theory. In the truncation (2.6), the bosonic part of the lagrangian (2.1) reduces to

$$\mathcal{L} = \frac{1}{4}\sqrt{G}R + \frac{1}{2}\sqrt{G}G^{\mu\nu}\,\partial_\mu Q\,\partial_\nu Q + V_Q\,, \tag{2.7}$$

with a scalar potential derived from a superpotential W as

$$V_Q = \frac{1}{2}\,(\partial_Q W)^2 - 2W^2\,, \qquad W \equiv -\frac{g}{8}\,\Big(13 + 20\cosh(\sqrt{2}Q) - \cosh(\sqrt{8}Q)\Big)\,. \tag{2.8}$$

With the standard domain wall ansatz

$$ds^2 = e^{2A(r)}\,\eta_{ij}\,dx^i dx^j + dr^2\,, \tag{2.9}$$

the field equations may be reduced to first-order form:

$$\partial_r Q = \partial_Q W\,, \qquad \partial_r A = -2\,W\,. \tag{2.10}$$

They may analytically be solved by

$$\frac{(5-y)(y+1)^2}{16\,(y-1)^3} = e^{24gr}\,, \qquad e^{6A(r)} = \frac{(5-y)^4}{128\,(y+1)(y-1)^6}\,, \qquad y = \cosh(\sqrt{2}Q)\,. \tag{2.11}$$

This solution interpolates between the origin and the supersymmetric extremum, preserving $N = (1,1)$ supersymmetry throughout the flow. From now on, we will fix the gauge coupling constant to the numerical value $g = 1/8$, thereby setting L_0, the AdS length at the origin of the scalar potential, to unity.

Let us emphasize that for the computation of correlation functions, it is indispensable to have an *exact*, meaning analytic, domain wall solution (2.11), rather than just finding the first-order equations and solving them numerically, which may be sufficient for other purposes. The point is that it is the fluctuations of supergravity fields *around the domain wall solution* that contain information on correlators in the boundary field theory. In particular, these second-order fluctuation equation are supplied with boundary conditions that specify the value of a supergravity field at the AdS boundary and demand a given (regular) behavior in the deep interior of the bulk space. The latter condition cannot be fixed perturbatively from the boundary, but is fixed only by solving the fluctuation equations. One might say this requires the "nonperturbative" bulk information that encodes the two-point function.[3] The fluctuation equations we obtain are highly singular at infinity, and it is not clear that one could have obtained our correlators even numerically if our domain wall solution would have amounted to just a numerical solution of the first-order equations.

[3]Of course, "perturbative" here refers to the radial expansion, which is a derivative (low-energy) expansion, rather than the coupling constant expansion in the field theory.

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3. Field equations and near boundary analysis

Thus, we embark on the way to computing correlators using the domain wall solution given above. As a first step we will derive the supergravity field equations and determine the coefficients in the near-boundary expansion of the supergravity fields. The truncated supergravity lagrangian is given in (2.7). In principle, it contains the complete information to compute correlation functions of the stress-energy tensor along the flow. Since we will be interested in computing correlation functions of the operators associated with the inert scalars as well, we extend (2.7) by expanding the original lagrangian (2.1) to second order in the inert scalars. This is sufficient for all further computations in this paper, as we will only treat correlation functions with at most two insertions of inert scalar operators. We denote the inert scalars collectively by

$$\Phi^{\mathbf{i}} = \left\{\Phi^{\mathbf{1}}, \Phi^{\mathbf{9}+}, \Phi^{\mathbf{9}-}, \Phi^{\mathbf{4}+}, \Phi^{\mathbf{4}-}\right\}, \tag{3.1}$$

where the index $\mathbf{i}$ denotes the dimension of the representations under $\mathrm{SO}(4)_{\mathrm{inv}}$ while the superscripts $\pm$ label the two-fold degeneracies, cf. the spectrum in table 1. Recall that the scalar fields in the $(\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3})$ appear in a multiplet together with the vector fields to which they are related by gauge symmetry. Hence, these scalars require a separate analysis and will not be treated in this paper.

The scalar fields parametrize an $\mathrm{SO}(8,n)$ matrix according to[4]

$$\mathcal{S} = \mathcal{S}_Q\, \mathcal{S}_\Phi\,, \qquad \text{with} \quad \mathcal{S}_\Phi = \exp \sum_{\mathbf{i}} \Phi^{\mathbf{i}}\, Y^{\mathbf{i}}\,, \tag{3.2}$$

where $\mathcal{S}_Q$ carries the entire dependence on the active scalar Q, and $Y^{\mathbf{i}}$ denote the noncompact generators of $\mathrm{SO}(8,n)$ associated with the representations of $\Phi^{\mathbf{i}}$, see [6, 5] for details. With this ansatz, the current in the kinetic term of (2.1) becomes

$$\mathcal{P}^{Ir}_\mu\, Y^{Ir} = \mathcal{S}^{-1}\, D_\mu \mathcal{S} = \mathcal{S}_\Phi^{-1}\, D_\mu \mathcal{S}_\Phi + \mathcal{S}_\Phi^{-1} \left(\mathcal{S}_Q^{-1} D_\mu \mathcal{S}_Q\right)\, \mathcal{S}_\Phi\,, \tag{3.3}$$

where the I, r indices belong to the vector representations of $\mathrm{SO}(8)$ and $\mathrm{SO}(n)$, respectively, and the relation to the $\mathbf{i}$ representation sector index is as described in section 2.

The near-boundary analysis is most conveniently performed in Fefferman-Graham coordinates $(x^i, \rho = e^{-2r})$, taking the metric to be

$$ds^2 = \frac{1}{4\rho^2}\, d\rho^2 + \frac{1}{\rho}\, g_{ij} dx^i dx^j \qquad i = 1, \ldots, d\,, \tag{3.4}$$

where for the moment we let d, the dimension of the boundary, be arbitrary for the sake of easy comparison with standard literature. The action on the asymptotically AdS space M is then of the general form

$$\begin{aligned} S = &\int_M d^d x\, d\rho \sqrt{G} \left(\frac{1}{2\kappa} R + \frac{1}{2} G^{\mu\nu} K(\Phi)\, \partial_\mu Q \partial_\nu Q + \frac{1}{2} G^{\mu\nu} \sum_{\mathbf{i}} \partial_\mu \Phi^{\mathbf{i}} \partial_\nu \Phi^{\mathbf{i}} + V(Q, \Phi)\right) \\ &- \frac{1}{2} \int_{\partial M} d^d x\, \sqrt{\gamma}\, \mathcal{K}\,, \end{aligned} \tag{3.5}$$

[4] To avoid confusion, we will always explicitly write out the sums over the different representations labelled by $\mathbf{i}$.

for general gravitational coupling constant κ. We denote by $\mathcal{K}$ the trace of the extrinsic curvature tensor $\mathcal{K}_{ij}$ on the hypersurface ∂M, and $\mathcal{K}_{ij}$ itself is given by

$$\mathcal{K}_{ij} = \frac{1}{\rho}\, g_{ij} - \partial_\rho g_{ij} \,, \tag{3.6}$$

in the metric (3.4). The induced metric on the hypersurface ∂M is denoted γ_{ij}. The functions $V(Q,\Phi)$ and $K(\Phi)$ in (3.5) are obtained from expanding (to second order in Φ) the scalar potential and the kinetic term, respectively, of (2.1).[5] Somewhat miraculously, the quadratic parts of the two functions may simultaneously be diagonalized with Q-independent eigenvectors in each of the two-fold degenerate representation sector **4** and **9**, respectively, and take the form

$$V(Q,\Phi) = V_Q + \sum_{\mathbf{i}} V_{\mathbf{i}}(Q)\, \Phi^{\mathbf{i}}\Phi^{\mathbf{i}} \,, \qquad K(\Phi) = 1 + \frac{1}{2}\sum_{\mathbf{i}} K_{\mathbf{i}}\, \Phi^{\mathbf{i}}\Phi^{\mathbf{i}} \,, \tag{3.7}$$

with

$$K_{\mathbf{1}} = K_{\mathbf{9}-} = K_{\mathbf{4}-} = 1 \,, \qquad K_{\mathbf{9}+} = K_{\mathbf{4}+} = 0 \,, \tag{3.8}$$

and the $V_{\mathbf{i}}(Q)$ given in (5.1), (5.2) below. The equations of motion from varying this action are, for a general metric $G_{\mu\nu}$ [6]

$$\begin{aligned}
R_{\mu\nu}[G] + \frac{2}{d-1}\,\Lambda G_{\mu\nu} &= -\kappa\Big[K\,\partial_\mu Q \partial_\nu Q + \sum_{\mathbf{i}} \partial_\mu \Phi^{\mathbf{i}} \partial_\nu \Phi^{\mathbf{i}} + \frac{2G_{\mu\nu}}{d-1}\,(V - V_{(0)})\Big] \,,\\
\Box_G\, \Phi^{\mathbf{i}} &= \frac{\partial}{\partial \Phi^{\mathbf{i}}}\left(V + \frac{1}{2} K G^{\mu\nu}\partial_\mu Q \partial_\nu Q\right) ,\\
\boxed{\scriptstyle K}_G\, Q &= \frac{\partial V}{\partial Q} \,,
\end{aligned} \tag{3.9}$$

where $V_{(0)} = V(Q,\Phi)|_{Q=0,\Phi^{\mathbf{i}}=0}$, and $\Lambda = \kappa V_{(0)} = -\frac{d(d-1)}{2}$, and $\boxed{\scriptstyle K}_G Q$ is the expression obtained by varying $\frac{1}{2}\sqrt{G} K G^{\mu\nu}\partial_\mu Q \partial_\nu Q$ and discarding a boundary term. With the metric (3.4), the Einstein equations can be written in the following form,

$$\begin{aligned}
&\rho[2g''_{ij} - 2(g' g^{-1} g')_{ij} + \mathrm{Tr}(g^{-1}g')g'_{ij}] + R_{ij}[g] - (d-2)\, g'_{ij} - \mathrm{Tr}(g^{-1}g')g_{ij} = \\
&\qquad\qquad = -\kappa\Big[K\,\partial_i Q \partial_j Q + \sum_{\mathbf{i}} \partial_i \Phi^{\mathbf{i}} \partial_j \Phi^{\mathbf{i}} + \frac{2}{(d-1)}\frac{g_{ij}}{\rho}\,(V - V_{(0)})\Big] \,,\\
&\nabla_i\, \mathrm{Tr}(g^{-1}g') - \nabla^j g'_{ij} = -2\kappa\,\Big[K\, Q' \partial_i Q + \sum_{\mathbf{i}} \Phi^{\mathbf{i}\,\prime} \partial_i \Phi^{\mathbf{i}}\Big] \,,\\
&\mathrm{Tr}(g^{-1}g'') - \frac{1}{2}\,\mathrm{Tr}(g^{-1}g'g^{-1}g') = -2\kappa\,\Big[K\,(Q')^2 + \sum_{\mathbf{i}} (\Phi^{\mathbf{i}\,\prime})^2 + \frac{V - V_{(0)}}{2(d-1)\rho^2}\Big] \,,
\end{aligned} \tag{3.10}$$

[5]It is due to our parametrization (3.2), (3.3) that the inert scalars automatically arise with a canonical kinetic term, whereas the kinetic term of the active scalar Q depends on the inert scalars. In contrast, the commonly used parametrization (which corresponds to choosing $\mathcal{S} = \mathcal{S}_\Phi \mathcal{S}_Q$ in (3.2)) gives a canonical kinetic term for the active scalar and a Q-dependent metric for the inert scalars. This requires additional rescaling of the inert scalars in order to diagonalize their equations of motion, cf. [23], which essentially amounts to reverting to the parameterization (3.2).

[6]Our curvature conventions are $R_{\mu\nu\sigma}{}^{\tau} = \partial_\mu \Gamma^\tau_{\nu\sigma} + \Gamma^\tau_{\mu\lambda}\,\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)$, $R_{\mu\nu} = R_{\mu\tau\nu}{}^{\tau}$.

and we note that here $R_{ij}[g]$ means the Ricci tensor for the d-dimensional metric g_{ij} only, i.e. only $R_{ikj}{}^{k}$ and no $R_{i\rho j}{}^{\rho}$ piece. The scalar equations take the form

$$
\begin{aligned}
&4\rho^2\Phi^{\mathbf{i}\,\prime\prime} + 2\rho\Phi^{\mathbf{i}\,\prime}\left[(2-d)+\rho\,(\log g)'\right] + \rho\,\Box_g\Phi^{\mathbf{i}} = \\
&\qquad\qquad\qquad = \left(2V_{\mathbf{i}} + \frac{1}{2}K_{\mathbf{i}}(4\rho^2 Q'Q' + \rho g^{ij}\partial_i Q\partial_j Q)\right)\Phi^{\mathbf{i}}\,, \\
&4\rho^2(KQ'' + K'Q') + 2\rho KQ'\left[(2-d)+\rho\,(\log g)'\right] + \rho\,\boxtimes_g Q - \partial_Q V = 0\,.
\end{aligned}
\tag{3.11}
$$

Primes here denote derivatives with respect to ρ, and $g = \det g_{ij}$. From now on, we specialize to $d=2$, but at times we will return to compare our results to those of higher dimensions, $d=4$ in particular. For reference, it will be useful to have the first Einstein equation for $g_{ij} = f(\rho)\,\eta_{ij}$:

$$
2\rho^2 f'' - 2\rho f' + 2\kappa\,(V - V_{(0)})\,f = 0\,, \tag{3.12}
$$

and the scalar equation of motion for x-independent $Q = Q(\rho)$ in this metric, with $\Phi^{\mathbf{i}} = 0$:

$$
4\rho^2\left(\frac{f'}{f}Q' + Q''\right) = \frac{\partial V}{\partial Q}\,. \tag{3.13}
$$

3.1 Perturbative solution

We will now solve the above equations of motion perturbatively for the first few terms in the expansion in ρ. First, we note that the kink solution (2.11) admits an expansion according to

$$
\begin{aligned}
Q_{\rm B} &= \rho^{1/4}\left(1 + \frac{1}{24}\sqrt{\rho} - \frac{13}{640}\rho + \cdots\right), \\
G_{{\rm B}\,ij} &= \rho^{-1}\eta_{ij}\left(1 - \sqrt{\rho} + \frac{7}{16}\rho + \cdots\right),
\end{aligned}
\tag{3.14}
$$

where "B" is for background. Consistency of these expansions can be conveniently checked using (3.12) and (3.13). The appearance of square roots of ρ in these expansions is generic in three bulk dimensions. One could have chosen to expand in a different variable to avoid the square roots, but we find it more useful to stay notationally close to the higher-dimensional literature. In general, the metric and scalars may also depend on x, and we expand the fields in ρ as

$$
\begin{aligned}
G_{ij}(x,\rho) &= \rho^{-1}g_{ij}(x,\rho) \\
&= \rho^{-1}\Big\{g_{(0)\,ij}(x) + \sqrt{\rho}g_{(1)\,ij}(x) + \rho\left(g_{(2)\,ij}(x) + h_{(2)\,ij}(x)\log\rho\right) + \cdots\Big\}\,, \\
Q(x,\rho) &= \rho^{1/4}q(x,\rho) = \rho^{1/4}\Big\{q_{(0)}(x) + \sqrt{\rho}\left(q_{(1)}(x) + \tilde{q}_{(1)}(x)\log\rho\right) + \cdots\Big\}\,, \\
\Phi(x,\rho) &= \rho^{1/4}\phi(x,\rho) = \rho^{1/4}\Big\{\phi_{(0)}(x) + \sqrt{\rho}\left(\phi_{(1)}(x) + \tilde{\phi}_{(1)}(x)\log\rho\right) + \cdots\Big\}\,.
\end{aligned}
\tag{3.15}
$$

The subscripts in parenthesis denote the order in $\sqrt{\rho}$, and also the highest number of derivatives with respect to x that will occur in these coefficients, hence this is a derivative (low-energy) expansion. The coefficients of the leading terms $g_{(0)\,ij}$, $q_{(0)}$, $\phi_{(0)}$ are interpreted as source terms for the dual operators. The field equations then determine the next few coefficients as algebraic functions of these boundary data. At a certain order — here, $g_{(2)}$ for the metric and $\phi_{(1)}$ for the scalars — the desired coefficient cancels out of the perturbative equations of motion, and remains undetermined; it is related to the one-point function of the dual operator. At this order, the ansatz is generalized to include a logarithmic term as we have done in (3.15), and the coefficient of this logarithmic term is determined instead of the one that cancelled out. In general, for example in the $d = 4$ Coulomb branch (CB) flow (see e.g. [11]), one may also need higher powers of logarithms in the ansatz, but we have checked that we do not.

The expansion of the scalar potential V_Q gives

$$V_Q = -\frac{1}{2} - \frac{3}{8}q_{(0)}\sqrt{\rho} - \frac{1}{8}\left(q_{(0)}^4 + 6q_{(0)}q_{(1)}\right)\rho - \frac{3}{4}q_{(0)}\tilde{q}_{(1)}\rho\log\rho + \cdots\,, \tag{3.16}$$

while for the inert scalar potentials $V_{\mathbf{i}}$ we find

$$V_{\mathbf{i}} = -\frac{3}{8} - \frac{1}{4}\left(1 + \frac{1}{4}K_{\mathbf{i}}\right)q_{(0)}^2\sqrt{\rho} + \cdots\,. \tag{3.17}$$

To this order, the potentials $V_{\mathbf{i}}$ come in just two different forms, depending on whether the value of $K_{\mathbf{i}}$ (3.8) is 0 or 1. It is only at higher order that the potentials $V_{\mathbf{i}}$ begin to differ between the various inert scalars. We can now expand the full potential as

$$V(Q,\Phi) = V_{(0)} + \sqrt{\rho}\,V_{(1)}(x) + \sqrt{\rho}\log\rho\,\tilde{V}_{(1)}(x) + \rho\,V_{(2)}(x) + \rho\log\rho\,\tilde{V}_{(2)}(x) + \cdots$$

and find for the coefficients

$$\begin{aligned}
V_{(0)} &= -\frac{1}{2}\,, \qquad V_{(1)} = -\frac{3}{8}\left(q_{(0)}^2 + \sum_{\mathbf{i}}(\phi_{(0)}^{\mathbf{i}})^2\right), \qquad \tilde{V}_{(1)} = 0\,,\\
V_{(2)} &= -\frac{1}{8}q_{(0)}^4 - \frac{1}{4}\sum_{\mathbf{i}}\left(1 + \frac{1}{4}K_{\mathbf{i}}\right)q_{(0)}^2(\phi_{(0)}^{\mathbf{i}})^2 - \frac{3}{4}\left(q_{(0)}q_{(1)} + \sum_{\mathbf{i}}\phi_{(0)}^{\mathbf{i}}\phi_{(1)}^{\mathbf{i}}\right),\\
\tilde{V}_{(2)} &= -\frac{3}{4}\left(q_{(0)}\tilde{q}_{(1)} + \sum_{\mathbf{i}}\phi_{(0)}^{\mathbf{i}}\tilde{\phi}_{(1)}^{\mathbf{i}}\right).
\end{aligned} \tag{3.18}$$

To lowest order, only the potential V_Q for the active scalar contributes. The fact that $\tilde{V}_{(1)}$ vanishes is required for consistency; there is a $1/\rho$ in front of V in the equation of motion, giving a total of $\rho^{-1/2}\log\rho$ for the $\tilde{V}_{(1)}$ term, but there is no term of that order to match it on the left-hand side of Einstein's equation.

3.2 Metric coefficients

Solving the (ij) component of the Einstein field equations (3.10) for each coefficient in (3.15) leads to[7]

$$\begin{aligned}
g_{(1)ij} &= \frac{4}{3}\kappa V_{(1)}\,g_{(0)ij}\,, \qquad \mathrm{Tr}\,g_{(2)} = \frac{1}{2}R[g_{(0)}] + 2\kappa V_{(2)} + \frac{8}{3}\kappa^2 V_{(1)}^2\,,\\
h_{(2)ij} &= \kappa\tilde{V}_{(2)}\,g_{(0)ij}\,,
\end{aligned} \tag{3.19}$$

[7] Here and in the following, we use the matrix notation $\mathrm{tr}\,A \equiv \mathrm{tr}(g_{(0)}^{-1}A)$ and $AB \equiv Ag_{(0)}^{-1}B$, i.e. indices are raised and lowered with the boundary metric $g_{(0)ij}$.

at orders $\rho^{-1/2}$, ρ^0, and $\log\rho$, respectively. When one arrives at order ρ^0 of the Einstein equation, which would determine the coefficient $g_{(2)ij}$, this coefficient only appears traced on the left-hand side. Thus, the non-trace part of $g_{(2)ij}$ remains undetermined in perturbation theory, which is expected as remarked above. Note that the last equation in (3.19) is understood to hold only up to terms in order $\phi^4_{(0)}$, since our starting action (3.5) was valid up to this order, cf. the discussion in the beginning of this section. It is further worth pointing out that, in contrast to the known $d = 4$ examples (e.g. the GPPZ flow), the $h_{(2)}$ coefficient in (3.19) is excited only by logarithmic terms in the scalars.

From the $(i\rho)$ component of Einstein's equations, one further derives

$$\begin{aligned}\nabla^j g_{(2)ij} =\ & \nabla_i\ \mathrm{Tr}\, g_{(2)} - \frac{3}{8}\nabla_i\ \mathrm{Tr}\, g^2_{(1)} - \frac{1}{4}g_{(1)ij}\nabla^j\ \mathrm{Tr}\, g_{(1)} + \\ & + \frac{1}{2}\nabla^j\,(g^2_{(1)ij}) + \frac{\kappa}{2}\left(q_{(0)}\nabla_i q_{(1)} + 3\,q_{(1)}\nabla_i q_{(0)}\right) + \\ & + \frac{\kappa}{2}\sum_{\mathbf{i}}\left(\phi^{\mathbf{i}}_{(0)}\nabla_i\phi^{\mathbf{i}}_{(1)} + 3\phi^{\mathbf{i}}_{(1)}\nabla_i\phi^{\mathbf{i}}_{(0)} + \frac{1}{2}K_{\mathbf{i}}(\phi^{\mathbf{i}}_{(0)})^2 q_{(0)}\nabla_i q_{(0)}\right). \end{aligned} \tag{3.20}$$

This expression will be used in section (4.5) to verify one of the Ward identities.

3.3 Scalar coefficients

The equations of motion for the inert scalar fields (3.11) in $d = 2$ read

$$2\rho^2\Phi^{\mathbf{i}\,\prime\prime} + \rho^2\,(\log g)'\Phi^{\mathbf{i}\,\prime} + \frac{1}{2}\rho\,\Box_g\Phi^{\mathbf{i}} = \left(V_{\mathbf{i}} + K_{\mathbf{i}}\left(\rho^2 Q'Q' + \frac{1}{4}\rho g^{ij}\partial_i Q\partial_j Q\right)\right)\Phi^{\mathbf{i}}. \tag{3.21}$$

Expanding the right-hand side of (3.21) in ρ, we find

$$-\frac{3}{4}\rho^{1/4}\left(\phi^{\mathbf{i}}_{(0)} + \left(\phi^{\mathbf{i}}_{(1)} + \frac{2}{3}\phi^{\mathbf{i}}_{(0)}q^2_{(0)}\right)\sqrt{\rho} + \tilde{\phi}^{\mathbf{i}}_{(1)}\sqrt{\rho}\log\rho + \cdots\right). \tag{3.22}$$

Remarkably, $K_{\mathbf{i}}$ drops out of the expression, i.e. to this order in ρ, the effective potentials of all the inert scalar fields coincide. Now expanding also the left-hand side of (3.21), we find that equating the most divergent terms (orders ρ^{-1} and $\rho^{-1/2}\log\rho$) on the two sides just gives consistency conditions. The next-to-leading-order nonlogarithmic terms $\rho^{-1/2}$ have the same dependence on $\phi_{(1)}$ on both sides, so it cancels out. With the logarithmic term in the ansatz (3.15), the coefficient $\tilde{\phi}_{(1)}$ also appears at order $\rho^{-1/2}$, but it does not cancel and is determined as promised:

$$\tilde{\phi}^{\mathbf{i}}_{(1)} = \frac{1}{3}\left(\mathrm{Tr}\, g_{(1)} + 2q^2_{(0)}\right)\phi^{\mathbf{i}}_{(0)} = \mathcal{O}(\phi^3)\,, \tag{3.23}$$

which upon using (3.19) is of only cubic order in the inert scalars, i.e. it vanishes to the order of validity of our computation for all the inert scalars. As anticipated above, the coefficient $\phi^{\mathbf{i}}_{(1)}$ remains undetermined by the field equations. For the active scalar Q, one derives from (3.11) the analogous equation

$$\tilde{q}_{(1)} = -\frac{1}{8}\left(\mathrm{Tr}\, g_{(1)} + 2q^2_{(0)} + 2\sum_{\mathbf{i}}(\phi^{\mathbf{i}}_{(0)})^2\right)q_{(0)} = \mathcal{O}(\phi^4)\,, \tag{3.24}$$

while $q_{(1)}$ remains undetermined as expected. We are now in a position to use these results to compute divergences of the on-shell action and, from there, counterterms and one-point functions. From now on, we set $\kappa = 2$ for the gravitational coupling constant.

4. Counterterms and one-point functions

4.1 Counterterms

As is well known, the on-shell action of gravity and scalars is a priori divergent even on a background of pure AdS. It contains divergences coming from the $\rho \to 0$ (boundary) side, and the standard regularization is to consider the action on a surface away from the boundary $\rho = \epsilon$ for some small coordinate distance ϵ. Then, divergences appear as poles and logarithms in ϵ that can be subtracted, which corresponds to some subtraction scheme in the boundary field theory. Finally, one can let $\epsilon \to 0$ to obtain the renormalized action. In addition to this standard procedure, we also want to ensure the on-shell action vanishes when evaluated on the background; this is necessary (but as we will see in section 5, not sufficient in our case) to make sure the renormalization scheme is supersymmetric [11, 12]. Since this formalism is not yet widely familiar, we will make an effort to explain all steps carefully. (In this section, for transparency of the intermediate expressions, we suppress the indices $\mathbf{i}$ labeling the representation sectors of inert scalars, and restore them at the end.) To start from the beginning, the first step of holographic renormalization is one of convenience: to eliminate the Einstein-Hilbert term in the bulk action against all scalar kinetic terms on-shell. This is an amusing exercise that can be performed in any dimension, simply by tracing the Einstein equation in (3.9) and substituting the Ricci scalar so obtained into the action (3.5). Only the potential V then remains in the bulk action, with the coefficient changed from $+1$ to -2, and there is a factor of $1/2$ from $\sqrt{G} = \sqrt{g}/2\rho^2$. Explicitly,

$$\begin{aligned} S_{\rm reg} = \int_{\rho\geq\epsilon} d^d x\, d\rho \sqrt{G}\, \Big(\frac{1}{2\kappa} R + \frac{1}{2} G^{\mu\nu} K(\Phi)\, \partial_\mu Q \partial_\nu Q + \\ + \frac{1}{2} G^{\mu\nu} \sum_{\mathbf{i}} \partial_\mu \Phi^{\mathbf{i}} \partial_\nu \Phi^{\mathbf{i}} + V(Q,\Phi) \Big) - \frac{1}{2} \int_{\rho=\epsilon} d^d x \sqrt{\gamma}\, \mathcal{K}\,, \qquad (4.1) \\ = - \int \int_\epsilon^{\rho_{\rm cr}} d^2 x\, d\rho\, \frac{\sqrt{g}}{\rho^2}\, V - \frac{1}{2} \int_{\rho=\epsilon} d^2 x \frac{\sqrt{g}}{\epsilon}\, \mathcal{K}\,, \qquad (4.2) \end{aligned}$$

where in the extrinsic curvature term we replaced $\sqrt{\gamma} = \sqrt{g}/\epsilon$. Although expressions (4.1) and (4.2) are on-shell equivalent, the action (4.2) is only convenient for determining counterterms and later we return to using (4.1). Here, the upper limit of integration $\rho_{\rm cr}(x)$ can be defined in general as giving the surface $\rho = \rho_{\rm cr}(x)$ of vanishing area, as in [24], but for a stationary metric as we consider here, all one needs is that the bulk lapse and shift functions vanish at $\rho = \rho_{\rm cr}$. In the coordinates (3.4), this simply means $\rho_{\rm cr} = \infty$.

One then performs an expansion in ϵ of (4.2). Gravity without scalars in three bulk dimensions is expected to give a constant volume divergence plus a finite term involving $\operatorname{Tr} g_{(2)}$. However, when we couple scalars we get all kinds of terms, including $(\log\rho)^2$ due to Φ^2, as we shall see. We will use the expansion of $\sqrt{\det g}$ to finite order:

$$\sqrt{g} = \sqrt{g_{(0)}} \left\{ 1 + \sqrt{\epsilon} \left(\frac{1}{2} \operatorname{Tr} g_{(1)} \right) + \epsilon \left(\frac{1}{2} \operatorname{Tr} g_{(2)} \right) + \epsilon \log\epsilon \left(\frac{1}{2} \operatorname{Tr} h_{(2)} \right) + \mathcal{O}(\epsilon^{3/2}) \right\}. \quad (4.3)$$

Notice that the combination $\operatorname{Tr} g_{(1)}^2 - \frac{1}{2}(\operatorname{Tr} g_{(1)})^2$ vanishes for our solution, since $g_{(1)ij}$ is simply proportional to $g_{(0)ij}$. Substituting in all expansions, and staying with a general potential, we arrive at

$$\begin{aligned} S_{\text{reg}} = \int d^2x \sqrt{g_{(0)}} \Big\{ & \epsilon^{-1}(-V_{(0)} - 1) + \epsilon^{-1/2} \left(-\operatorname{Tr} g_{(1)} V_{(0)} - 2V_{(1)} - \frac{1}{4} \operatorname{Tr} g_{(1)} \right) + \\ & + \log \epsilon \left(\frac{1}{2} \operatorname{Tr} g_{(2)} V_{(0)} + \frac{1}{2} \operatorname{Tr} g_{(1)} V_{(1)} + V_{(2)} - \frac{d-2}{4} \operatorname{Tr} h_{(2)} \right) + \\ & + (\log \epsilon)^2 \frac{1}{4} \left(\tilde{V}_{(2)} + \frac{1}{2} \operatorname{Tr} h_{(2)} \tilde{V}_{(0)} \right) + \\ & + \epsilon^0 \left(-\frac{d-2}{4} \operatorname{Tr} g_{(2)} \right) + \mathcal{O}(\epsilon^{1/2}) \Big\} . \end{aligned} \tag{4.4}$$

Of course we consider $d = 2$, but we found it useful to keep coefficients of the form $(d-2)$ to display some cancellations. In particular, we see that the finite $\operatorname{Tr} g_{(2)}$ contribution from the extrinsic curvature is cancelled. Substituting our perturbative solution for the metric (eq. (3.19)) into (4.4), we stumble upon some pleasing simplifications:

$$S_{\text{reg}} = \int d^2x \sqrt{g_{(0)}} \left\{ \epsilon^{-1} \left(-\frac{1}{2} \right) + \epsilon^{-1/2} \left(-\frac{2}{3} V_{(1)} \right) + \log \epsilon \left(-\frac{1}{8} R \right) + \mathcal{O}(\epsilon^{1/2}) \right\} . \tag{4.5}$$

From a pure supergravity point of view, the cancellations in the $\log \epsilon$ term seem quite surprising: the back reaction of the metric to the scalars exactly cancel the explicit contributions of the scalars and leaves only the curvature. Also in the GPPZ flow [11], there is no new "cross-term" anomaly, even though in that case as well as here, fluctuations of the active scalar and metric are coupled. In fact, holography provides a simple explanation for this. The boundary expectation value $\langle T_i^i \rangle$, with scalar sources set to zero, is the trace of the variation of the generating functional in the boundary field theory with respect to to the boundary metric $g_{(0)ij}(x)$ — also with scalar sources set to zero. Hence, gravitational and scalar anomalies on the boundary can be separately computed, and must therefore simply add. On the other hand, one could have expected a separate scalar matter anomaly here like the $\Phi \Box_\gamma \Phi + \frac{1}{6} R[\gamma] \Phi^2$ in four dimensions, but with $R[\gamma] = \epsilon R[g]$ and $\Box_\gamma = \epsilon \Box_g$ we see that these terms will vanish as $\epsilon \to 0$ in our case.

We now proceed to regularize the action with local covariant counterterms as outlined in [11]. The philosophy is the following: first, compute the divergent part of the regularized action S_{reg} for a given potential. Second, pull back all quantities to the regulating surface $\rho = \epsilon$, i.e. express the coefficients $q_{(n)}$ in terms of Q, and analogously for the inert scalars. The full fields $Q(\rho, x)$ and $\Phi(\rho, x)$ are covariant, not the individual coefficients in the ρ expansion. Third, the covariant counterterm action $S_{\text{ct}}(Q, \Phi)$ is defined by having the same divergent parts as $S_{\text{reg}}(Q, \Phi)$ already obtained, *and* causing the final renormalized action

$$S_{\text{ren}} = \lim_{\epsilon \to 0} (S_{\text{reg}} + S_{\text{ct}}) \tag{4.6}$$

to vanish on the background, which implies $\langle T_{ij} \rangle = 0$.

We now perform these steps in the present setting. For the inert scalars, substituting the $V_{(1)}$ given in (3.18) in the $\epsilon^{-1/2}$ term, we find

$$S_{\text{reg}} = \int d^2x \sqrt{g_{(0)}} \Big(-\frac{1}{2\epsilon} + \frac{1}{4\sqrt{\epsilon}} (q_{(0)}^2 + \phi_{(0)}^2) - \log \epsilon \, \frac{1}{8} R[g_{(0)}] + \mathcal{O}(\epsilon^{1/2}) \Big) . \tag{4.7}$$

To pull this back to $\rho = \epsilon$ we need the inverse of (4.3):

$$\sqrt{g_{(0)}} = \sqrt{g} \left\{ 1 - \sqrt{\epsilon} \frac{1}{2} \operatorname{Tr} g_{(1)} + \epsilon \left[\frac{1}{4} \left(\operatorname{Tr} g_{(1)} \right)^2 - \frac{1}{2} \operatorname{Tr} g_{(2)} \right] - \epsilon \log \epsilon \frac{1}{2} \operatorname{Tr} h_{(2)} + \mathcal{O}(\epsilon^{3/2}) \right\} . \tag{4.8}$$

Substituting (4.8) in (4.7) and again using (3.19), we find

$$\begin{aligned} S_{\text{reg}} &= \int d^2x \sqrt{g} \Big(-\frac{1}{2\epsilon} - \frac{1}{4\sqrt{\epsilon}} (q_{(0)}^2 + \phi_{(0)}^2) + \qquad\qquad\qquad (4.9) \\ &\qquad + \epsilon^0 \left(\frac{1}{8} (q_{(0)}^4 + \phi_{(0)}^4) - \frac{3}{4} \phi_{(0)} \phi_{(1)} + \frac{1}{8} R[g_{(0)}] \right) - \log \epsilon \, \frac{1}{8} R[g_{(0)}] + \mathcal{O}(\epsilon^{1/2}) \Big) . \end{aligned}$$

(Notice the sign switch of the $1/\sqrt{\epsilon}$ term.) If we would add noncovariant counterterms to cancel only the divergences and then differentiate the remaining finite action, as one essentially did in the "old" prescription, we would find the wrong one-point function. What is wrong about it will be easier to see when we actually have the one-point function at hand, e.g. (4.21) for the active scalar.

Instead, in holographic renormalization we concentrate on the divergences in the covariant pulled-back action:

$$S_{\text{reg}} = \int d^2x \sqrt{g} \left(-\frac{1}{2\epsilon} - \frac{1}{\epsilon} \left(\frac{1}{4} \Phi^2 + \frac{1}{4} Q^2 \right) - \log \epsilon \, \frac{1}{8} R[g] + \mathcal{O}(\epsilon^0) \right) . \tag{4.10}$$

We remind the reader that it is $\phi_{(0)}$ and $g_{(0)}$ which are kept fixed as $\epsilon \to 0$, hence e.g. $Q^2 \sim \epsilon^{1/2}$ as $\epsilon \to 0$. Equation (4.10) determines the divergent parts of the covariant counterterm action S_{ct}. In total, including an overall minus sign and using $\sqrt{g} R[g] = \sqrt{\gamma} R[\gamma]$, the full $2d$ counterterm action is

$$S_{\text{ct}} = \int d^2x \sqrt{\gamma} \left(\frac{1}{2} + \frac{1}{4} Q^2 + \frac{1}{4} \sum_{\mathbf{i}} (\Phi^{\mathbf{i}})^2 + a Q^4 + \sum_{\mathbf{i}} b_{\mathbf{i}} \, Q^2 (\Phi^{\mathbf{i}})^2 + \log \epsilon \, \frac{1}{8} R[\gamma] \right) , \tag{4.11}$$

where the index $\mathbf{i}$ on Φ has been restored. We did not only cancel the divergences in (4.10), we also added the smallest set of finite counterterms needed for the renormalization scheme to preserve supersymmetry; the coefficients a and $b_{\mathbf{i}}$ are to be fixed.

A few comments about the counterterms are in order. First, the most divergent term in the action is the volume divergence, which is cancelled by a counterterm of $1/2$, or $(d-1)/2L$ in general d and for arbitrary AdS scale L. This counterterm has been known since the early days of the AdS/CFT correspondence, for general d it was introduced in [25]. The quadratic counterterm $\frac{1}{4}(\Phi^{\mathbf{i}})^2$ has also been familiar almost since the beginning. In [5] the coefficient of this term was computed in the fixed-background formalism for our case, and we indeed see it has the same coefficient (in general, $(d-\Delta)/2$) as in the fixed-background case, justifying that treatment in retrospect. A finite Q^4 counterterm was first displayed in [11], and the $Q^2(\Phi^i)^2$ terms can be seen as natural generalizations of the Q^4 term, with the additional complication that the coefficients $b^{\mathbf{i}}$ cannot be determined by evaluation on the background, since $\Phi^{\mathbf{i}}_{\mathrm{B}} = 0$. This is not a problem; in section 5.3 we will see how to compute $b^{\mathbf{i}}$, using a supersymmetry Ward identity for the two-point functions of superpartner inert scalars. The result is listed in table 2.

Notice we could not add trilinear terms like $Q(\Phi^{\mathbf{i}})^2$ since they would ruin the divergence structure; four powers of $\epsilon^{1/4}$ scalars are needed to reach finiteness. One could have added a finite counterterm proportional to the curvature scalar R, but as is well known, $\int d^2x\sqrt{g}R$ is a topological invariant in two dimensions (equal to 4π times the Euler number) hence the variation of this term with respect to the metric g_{ij} is zero, so it would not contribute to correlators.

Finally, we fix the coefficient a as in [11] by evaluating $S_{\mathrm{ren}} = \lim_{\epsilon\to 0}(S_{\mathrm{reg}} + S_{\mathrm{ct}})$ on the background:

$$\begin{aligned} S_{\mathrm{ren,B}} &= \int d^2x \left(a q^4_{(0)B} + \frac{1}{8}\,\mathrm{tr}(g_{(1)B}) q^2_{(0)B} + \frac{1}{2} q_{(0)B} q_{(1)B} + \frac{1}{4}\,\mathrm{tr}(g_{(2)B}) \right) \\ &= \int d^2x \left(a - \frac{1}{96} \right), \end{aligned} \tag{4.12}$$

where we used the background expansions (3.14). Hence $a = 1/96$, and we see that the naive subtraction of finite counterterms directly in the noncovariant action would have produced a different result. There is a useful check of the counterterm coefficients for the active scalar (see e.g. [26]): using the BPS equations, the expansion of the negative of $W(Q)$ gives the coefficients directly, as $-W(Q) = \frac{1}{2} + \frac{1}{4}Q^2 + \frac{1}{96}Q^4 + \cdots$.

4.2 One-point functions: generalities

Now that we have the renormalized action, we can apply the formula at the heart of the AdS/CFT correspondence in the supergravity approximation:

$$\underbrace{\langle\, e^{-\int d^2x\, \sqrt{g}\,\left(q_{(0)}\mathcal{O}_q + \phi^{\mathbf{i}}_{(0)}\mathcal{O}^{\mathbf{i}}_\phi\right)}\,\rangle_{g_{(0)ij}}}_{\text{field theory}} = \underbrace{e^{-S_{\mathrm{ren}}(Q,\Phi)}}_{\text{supergravity}} , \tag{4.13}$$

where the quantities on the left are the scaled Dirichlet data, e.g. q as opposed to Q.

The variation of the renormalized action for small variations in the sources $q_{(0)}$, $\phi_{(0)}$ and $g_{(0)}$ produces the boundary one-point functions:

$$\delta S_{\mathrm{ren}} = \int d^2x\sqrt{g_{(0)}}\left(\frac{1}{2}\langle T_{ij}\rangle \delta g^{ij}_{(0)} + \langle \mathcal{O}_q\rangle \delta q_{(0)} + \sum_{\mathbf{i}} \langle \mathcal{O}^{\mathbf{i}}_\phi\rangle \delta\phi^{\mathbf{i}}_{(0)}\right). \tag{4.14}$$

Since bulk diffeomorphisms correspond to global symmetries of the boundary theory, one should require these boundary quantities to satisfy Ward identities. Following [11], one can derive Ward identities for the one-point functions of the stress-energy tensor and scalars in the presence of sources:

$$\nabla^i \langle T_{ij}\rangle = -\langle \mathcal{O}_q\rangle \nabla_j q_{(0)} - \sum_{\mathbf{i}} \langle \mathcal{O}^{\mathbf{i}}_\phi\rangle \nabla_j \phi^{\mathbf{i}}_{(0)} , \tag{4.15}$$

$$
\begin{aligned}
\langle T_i^i \rangle &= (\Delta_q - 2) q_{(0)} \langle \mathcal{O}_q \rangle + \sum_{\mathbf{i}} (\Delta_{\phi^{\mathbf{i}}} - 2)\, \phi^{\mathbf{i}}_{(0)} \, \langle \mathcal{O}^{\mathbf{i}} \rangle + \mathcal{A} \\
&= -\frac{1}{2} q_{(0)} \, \langle \mathcal{O}_q \rangle - \frac{1}{2} \sum_{\mathbf{i}} \phi^{\mathbf{i}}_{(0)} \, \langle \mathcal{O}^{\mathbf{i}} \rangle + \mathcal{A} \,,
\end{aligned} \tag{4.16}
$$

where $\mathcal{A}$ is the conformal anomaly we saw arise in the previous section, from a logarithmic counterterm breaking radial bulk diffeomorphisms. If the reader finds this form of Ward identities unfamiliar, it is probably because standard quantum field theory Ward identities are usually expressed with sources set to zero, so that e.g. $\nabla^i \langle T_{ij} \rangle = 0$.

Explicitly, the anomaly $\mathcal{A}$ arises under a radial rescaling $\epsilon \to \mu^2 \epsilon$; all terms in S_{ren} above are manifestly invariant except the logarithmic term. It contributes an anomaly[8] $-\frac{1}{2} \log \mu^2 \mathcal{A}$, hence we identify $\mathcal{A} = -\frac{1}{4} R$. This is as was to be expected by holography; the conformal anomaly in 2d field theory on a space of scalar curvature R is simply proportional to R,

$$
-\frac{1}{4} R = -\frac{L}{2\kappa} R = -\frac{c}{24\pi} R = \langle T_i^i \rangle
$$

with $c = 3L/2G_N$ the Brown-Henneaux central charge and the gravitational coupling temporarily restored to $\kappa = 8\pi G_N$. The Ward identity (4.16), including this anomaly, will provide a useful check for the one-point function of T_{ij} computed below.

4.3 Inert scalars

The one-point function for the operators dual to inert scalars is now easy to compute, there is one contribution from the regularized action and some from the counterterms. It is convenient to also introduce the intermediate *subtracted* action $S_{\mathrm{sub}}(\epsilon) = S_{\mathrm{reg}}(\epsilon) + S_{\mathrm{ct}}(\epsilon)$, which is the quantity that becomes the renormalized action S_{ren} when we take ϵ to zero: $S_{\mathrm{ren}} = \lim_{\epsilon \to 0} S_{\mathrm{sub}}(\epsilon)$. The contribution from the regularized action (4.1) is

$$
\delta S_{\mathrm{reg}} = -2 \int_{\rho = \epsilon} d^2 x \sqrt{\gamma}\, \epsilon \sum_{\mathbf{i}} \delta \Phi^{\mathbf{i}} \partial_\epsilon \Phi^{\mathbf{i}}
$$

where we used the Φ bulk field equation (3.9) and $\sqrt{g} = \epsilon \sqrt{\gamma}$. Then, using the counterterm action S_{ct} from eq. (4.11) we can write down the functional derivative

$$
\begin{aligned}
\frac{1}{\sqrt{\gamma}} \frac{\delta S_{\mathrm{sub}}}{\delta \Phi^{\mathbf{i}}} &= -2\epsilon \partial_\epsilon \Phi^{\mathbf{i}} + \frac{1}{2} \Phi^{\mathbf{i}} + 2 b_{\mathbf{i}}\, Q^2 \Phi^{\mathbf{i}} \\
&= \epsilon^{1/4} \Bigg[-\frac{1}{2} \phi^{\mathbf{i}}_{(0)} + \frac{1}{2} \phi^{\mathbf{i}}_{(0)} + \epsilon^{1/2} \left(-\frac{3}{2} \phi^{\mathbf{i}}_{(1)} + \frac{1}{2} \phi^{\mathbf{i}}_{(1)} - \tilde{\phi}^{\mathbf{i}}{}_{(1)} + 2 b_{\mathbf{i}} q^2_{(0)} \phi^{\mathbf{i}}_{(0)} \right) + \\
&\qquad + \epsilon^{1/2} \log \epsilon \left(-\frac{3}{2} \tilde{\phi}^{\mathbf{i}}{}_{(1)} + \frac{1}{2} \tilde{\phi}^{\mathbf{i}}{}_{(1)} \right) + \cdots \Bigg] .
\end{aligned}
$$

This is divided by $\epsilon^{\Delta/2} = \epsilon^{3/4}$ and the limit $\epsilon \to 0$ is taken to yield the one-point function in the presence of sources

$$
\langle \mathcal{O}_{\phi^{\mathbf{i}}} \rangle = -(\phi^{\mathbf{i}}_{(1)} + \tilde{\phi}^{\mathbf{i}}_{(1)}) + 2 b_{\mathbf{i}}\, q^2_{(0)} \phi^{\mathbf{i}}_{(0)} = -\phi^{\mathbf{i}}_{(1)} + 2 b_{\mathbf{i}}\, q^2_{(0)} \phi^{\mathbf{i}}_{(0)} \,, \tag{4.17}
$$

[8]The factor $-1/2$ is standard, cf. [11, (5.43)].

where we emphasize that this only holds to linear order in $\phi^{\mathbf{i}}$, since we have only included $\Phi^{\mathbf{i}}$ up to quadratic order in the action. The forefactor of $\phi_{(1)}$ is -1, or $-(2\Delta - d)$ in general. (Notice that this is in [11] conventions; there is an overall sign switch in the one-point function from [10]). This forefactor also reproduces the one obtained in [5] using the fixed-background formalism, where the finite counterterm that appears here was neglected.

4.4 Active scalar

For the active scalar, the contribution of the regularized action (4.1) is given by

$$\delta S = \int_{\rho=\epsilon} d^2x \sqrt{\gamma}\, \delta Q \Big(-2\epsilon\, \partial_\epsilon Q - \epsilon \sum_{\mathbf{i}} K_{\mathbf{i}} \Phi^{\mathbf{i}} \Phi^{\mathbf{i}} \partial_\epsilon Q \Big)\,, \tag{4.18}$$

where we used the bulk field equation for Q. Proceeding as above, we obtain for the one-point function

$$\langle \mathcal{O}_q \rangle = -q_{(1)} + 4a q_{(0)}^3 + q_{(0)} \sum_{\mathbf{i}} (2b_{\mathbf{i}} - \frac{1}{4} K_{\mathbf{i}})\, \phi^{\mathbf{i}}_{(0)} \phi^{\mathbf{i}}_{(0)}\,. \tag{4.19}$$

Expanding around the background (3.14) according to

$$q_{(i)} = q_{\mathrm{B}(i)} + \varphi_{(i)}\,, \tag{4.20}$$

we see explicitly that $\langle \mathcal{O}_q \rangle$ vanishes on the background for $a = 1/96$ and there remain only the fluctuations:

$$\langle \mathcal{O}_q \rangle = \frac{1}{8} \varphi_{(0)} - \varphi_{(1)} - \frac{1}{4}\, \varphi_{(0)} \sum_{\mathbf{i}} \left(2b_{\mathbf{i}} - \frac{1}{4} K_{\mathbf{i}} \right) \phi^{\mathbf{i}}_{(0)} \phi^{\mathbf{i}}_{(0)} + \mathcal{O}(\varphi^2)\,. \tag{4.21}$$

This confirms that the flow is a true operator deformation; the boundary operator that drives the flow has vanishing vacuum expectation value. Notice that if we had not subtracted the finite counterterm aQ^4 in the action, or if we had tried to subtract a finite noncovariant counterterm $aq^4_{(0)}$ directly, we would have obtained a nonvanishing result for the one-point function $\langle \mathcal{O}_q \rangle$; this would have been an apparent contradiction with the claim that the deformation has vanishing vev.

4.5 Stress-energy tensor

We proceed to compute the stress-energy tensor one-point function in the same way: functionally differentiate the renormalized action with respect to the induced metric γ^{ij} on the surface $\rho = \epsilon$, but this computation is a little more complicated. It is convenient to split the computation into two parts, one corresponding to the extrinsic curvature, and one for the counterterms. The gravitational contribution due to the extrinsic curvature term is[9]

$$\begin{aligned} T_{\mathrm{reg,grav},ij} &= \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta \gamma^{ij}} \Big(-\frac{1}{2} \int_{\rho=\epsilon} d^2x \sqrt{\gamma}\, \mathcal{K} \Big) = -\frac{1}{2} \left(\mathcal{K}_{ij} - \mathcal{K} \gamma_{ij} \right) \\ &= -\frac{1}{2} \left(-\partial_\epsilon g_{ij} + g_{ij} g^{kl} \partial_\epsilon g_{kl} - \frac{d-1}{\epsilon} g_{ij} \right) \end{aligned}$$

[9] There is a sign mistake in [11, (4.11)]. The correct expression is given in [10, (3.6)].

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for general d, where by $\partial_\epsilon g_{ij}$ one intends $\partial_\rho g_{ij}(\rho,x)|_{\rho=\epsilon}$. Here the factor of 2 in the first expression comes from (4.14), but it immediately cancels with the 1/2 from the variation of $\sqrt{\gamma}$. For our metric ansatz we obtain

$$\begin{aligned}
T_{\text{reg,grav},\,ij} \;=\; -\frac{1}{2}\Bigg[&\epsilon^{-1}(-g_{(0)ij}) + \epsilon^{-1/2}\left(-\frac{3}{2}g_{(1)ij} + \frac{1}{2}g_{(0)ij}\operatorname{Tr} g_{(1)}\right) + \\
&+\epsilon^{0}\left(-2g_{(2)ij} - h_{(2)ij} + \frac{1}{2}g_{(1)ij}\operatorname{Tr} g_{(1)} - \frac{1}{2}g_{(0)ij}\operatorname{Tr} g_{(1)}^{2} + g_{(0)ij}\operatorname{Tr} g_{(2)}\right) + \\
&+\log\epsilon(g_{(0)ij}\operatorname{Tr} h_{(2)} - 2h_{(2)ij}) + \mathcal{O}(\epsilon^{1/2})\Bigg]\,,
\end{aligned}$$

for the gravitational part. The counterterm action contributes

$$\begin{aligned}
T_{\text{ct},ij} &= \frac{2}{\sqrt{\gamma}}\frac{\delta S_{\text{ct}}}{\delta\gamma^{ij}} = -\gamma_{ij}\left(\frac{1}{2} + \frac{1}{4}\sum_{\mathbf{i}}(\Phi^{\mathbf{i}})^2 + \frac{1}{4}Q^2 + aQ^4 + \sum_{\mathbf{i}} b_{\mathbf{i}}Q^2(\Phi^{\mathbf{i}})^2\right) \\
&= -\frac{1}{2}\,\epsilon^{-1}\,g_{(0)ij} - \frac{1}{2}\,\epsilon^{-1/2}\left[g_{(1)ij} + g_{(0)ij}\left(q_{(0)}^2 + \sum_{\mathbf{i}}(\phi^{\mathbf{i}}_{(0)})^2\right)\right] - \\
&\quad -\epsilon^{0}\left[\frac{1}{2}g_{(2)ij} + \frac{1}{4}\,g_{(1)ij}\left(q_{(0)}^2 + \sum_{\mathbf{i}}(\phi^{\mathbf{i}}_{(0)})^2\right) + \frac{1}{2}\,g_{(0)ij}\left(q_{(0)}q_{(1)} + \sum_{\mathbf{i}}\phi^{\mathbf{i}}_{(0)}\phi^{\mathbf{i}}_{(1)}\right) + \right. \\
&\qquad \left. + g_{(0)ij}\left(aq_{(0)}^4 + \sum_{\mathbf{i}} b_{\mathbf{i}}\,q_{(0)}^2(\phi^{\mathbf{i}}_{(0)})^2\right)\right] - \\
&\quad -\frac{1}{2}\,\log\epsilon\left[h_{(2)ij} + g_{(0)ij}\left(q_{(0)}\tilde{q}_{(1)} + \sum_{\mathbf{i}}\phi^{\mathbf{i}}_{(0)}\tilde{\phi}^{\mathbf{i}}{}_{(1)}\right) + \mathcal{O}(\epsilon^{1/2})\right],
\end{aligned}$$

where the logarithmic term is scheme-dependent, being due to the variation of any matter conformal anomaly. Here, using the results of sections 3.2 and 3.3, all terms at order $\log\epsilon$ above actually vanish, but we will keep them since the expressions may be useful in situations where they do not vanish. Also, the Ricci scalar R does not contribute, as mentioned above.

Putting everything together as $T_{\text{sub}} = T_{\text{reg,grav}} + T_{\text{ct}}$, the singular terms vanish upon using the earlier perturbative expressions. The finite part is the boundary one-point function, it is given by

$$\begin{aligned}
\langle T_{ij}\rangle &= \lim_{\epsilon\to 0} T_{\text{sub},\,ij} \qquad (4.22)\\
&= \frac{1}{2}g_{(2)ij} + \frac{1}{2}h_{(2)ij} - \frac{1}{4}g_{(1)ij}\left[\operatorname{Tr} g_{(1)} + (q_{(0)}^2 + \sum_{\mathbf{i}}(\phi^{\mathbf{i}}_{(0)})^2)\right] + \\
&\quad + g_{(0)ij}\left[\frac{1}{4}\operatorname{Tr} g_{(1)}^2 - \frac{1}{2}\operatorname{Tr} g_{(2)} - \frac{1}{2}q_{(1)}q_{(0)} - aq_{(0)}^4 - \sum_{\mathbf{i}}\frac{1}{2}\left(\phi^{\mathbf{i}}_{(1)} + 2b_{\mathbf{i}}\,q_{(0)}^2\phi^{\mathbf{i}}_{(0)}\right)\phi^{\mathbf{i}}_{(0)}\right].
\end{aligned}$$

On the background this evaluates to

$$\langle T_{ij}\rangle_{\text{B}} = \left(\frac{1}{96} - a\right)\eta_{ij} = 0\,,$$

which vanishes by the value previously determined for a. This is, of course, a trivial consequence of the fact that a was defined to cause S_{ren} to vanish on the background.

Using (3.20) one may verify after some computation that (4.22) yields

$$\nabla^j \langle T_{ij} \rangle = \left(q_{(1)} - 4aq_{(0)}^3 - \sum_{\mathbf{i}} \left(2b_{\mathbf{i}} - \frac{1}{4} K_{\mathbf{i}} \right) \phi_{(0)}^{\mathbf{i}} \phi_{(0)}^{\mathbf{i}} q_{(0)} \right) \nabla_i q_{(0)} + \\ + \sum_{\mathbf{i}} (\phi_{(1)}^{\mathbf{i}} - 2b_{\mathbf{i}}\, q_{(0)}^2 \phi_{(0)}^{\mathbf{i}})\, \nabla_i \phi_{(0)}^{\mathbf{i}}\,, \tag{4.23}$$

and thus satisfies the Ward identity (4.15) with the scalar one-point functions (4.17), (4.19). Similarly, tracing (4.22) and using the perturbative expressions for $\mathrm{Tr}\, g_{(1)}$ and $\mathrm{Tr}\, g_{(2)}$, we obtain

$$\langle T_i^i \rangle = -\frac{1}{4} R[g_{(0)}] + \frac{1}{2} \left(q_{(1)} - 4aq_{(0)}^3 \right) q_{(0)} + \sum_{\mathbf{i}} \frac{1}{2} \left(\phi_{(1)}^{\mathbf{i}} - \left(4b_{\mathbf{i}} - \frac{1}{4} K_{\mathbf{i}} \right) q_{(0)}^2 \phi_{(0)}^{\mathbf{i}} \right) \phi_{(0)}^{\mathbf{i}}\,, \tag{4.24}$$

which together with (4.17), (4.19), guarantees the Ward identity (4.16) including the conformal anomaly $\mathcal{A} = -\frac{1}{4}R$. Now we have our full collection of one-point functions and proceed to compute two-point functions.

5. Two-point functions of inert scalars

5.1 Fluctuation equations and the existence of prepotentials

As has been emphasized above, near-boundary analysis is no longer sufficient when we move on to the computation of 2-point functions. It needs to be supplemented with a solution to the equations of motion linearized around the background (2.11). For the inert scalars, it follows from (3.11) that these fluctuation equations turn into a three-dimensional Laplace equation in the domain wall metric, with total potential

$$V_{\mathbf{i}}^{\mathrm{tot}} = V_{\mathbf{i}} + \frac{1}{4} K_{\mathbf{i}} \, (\partial_Q W)^2 \Big|_{Q=Q_{\mathrm{B}}}\,. \tag{5.1}$$

These total potentials were computed in [5]:

$$\begin{aligned} V_{\mathbf{1}}^{\mathrm{tot}} &= \frac{1}{1024} \left(-45 - 160\, y + 10\, y^2 + 3\, y^4 \right), \\ V_{\mathbf{9}+}^{\mathrm{tot}} &= -\frac{1}{16} \left(17 + 30\, y + y^2 \right), \\ V_{\mathbf{9}-}^{\mathrm{tot}} &= \frac{1}{1024} \, (y+1)(-93 + 13\, y - 19\, y^2 + 3\, y^3)\,, \\ V_{\mathbf{4}+}^{\mathrm{tot}} &= -\frac{1}{16} \, (3+y)(7+5y)\,, \\ V_{\mathbf{4}-}^{\mathrm{tot}} &= \frac{1}{1024} \, (y+1)(y-5)(17 + 4\, y + 3\, y^2)\,, \end{aligned} \tag{5.2}$$

with $y = \cosh(\sqrt{2} Q_{\mathrm{B}})$. Further, it was shown in [5] that the resulting equations of motion can be transformed into one-dimensional Laplace equations in flat space, with effective potentials $\mathcal{V}_{\mathbf{i}}$ derived from prepotentials (in the sense of supersymmetric quantum mechanics) as $\mathcal{V}_{\mathbf{i}} = \mathcal{U}_{\mathbf{i}}' + \mathcal{U}_{\mathbf{i}}^2$. This underlying structure is crucial, as the absence of tachyonic fluctuations is then manifest. Although it is easy to see that $\mathcal{V}$ can always be rewritten in terms of a prepotential $\mathcal{U}$ for scalars with vanishing explicit potential $V^{\rm tot}$ (i.e. when the effective potential $\mathcal{V}$ is only due to the e^{2A} of the curved background, cf. (5.6) below), a priori it seems surprising that it would be possible for all our inert scalars with the various potentials above. A similar situation was observed in [16] for the active scalar fluctuations in the most prominent exact five-dimensional flows [3, 4, 27], which seemed somewhat puzzling and in need of explanation.

We now give a general argument for the existence of these prepotentials in the fluctuation equations of gauged supergravity and show how they may be directly extracted from the supergravity lagrangian. Although we stay with our model as a concrete example, the argument straightforwardly translates to other supergravities and higher dimensions. The fluctuations of inert scalars around the background solution $\Phi^{\mathbf{i}}$ are described by the bosonic lagrangian

$$\mathcal{L}^{\mathbf{i}} = \frac{1}{2}\sqrt{G_{\rm B}}\, G_{\rm B}^{\mu\nu}\partial_\mu\Phi^{\mathbf{i}}\partial_\nu\Phi^{\mathbf{i}} + \sqrt{G_{\rm B}}\, V_{\mathbf{i}}^{\rm tot}(Q_{\rm B})\,\Phi^{\mathbf{i}}\Phi^{\mathbf{i}}\,, \tag{5.3}$$

obtained from (3.5) upon evaluation on the background. We change to horospheric coordinates[10]

$$ds^2 = e^{2A_{\rm B}(\zeta)}\,(\eta_{ij}\,dx^i dx^j + d\zeta^2)\,, \qquad \text{i.e.}\quad \frac{d\zeta}{dr} = e^{-A_{\rm B}}\,. \tag{5.4}$$

Redefining $\Phi^{\mathbf{i}} = e^{-A_{\rm B}/2}R^{\mathbf{i}}$ and dropping a total derivative, this lagrangian takes the form

$$\mathcal{L}^{\mathbf{i}} = \frac{1}{2}\,\partial_\mu R^{\mathbf{i}}\partial^\mu R^{\mathbf{i}} + \frac{1}{2}\,\mathcal{V}_{\mathbf{i}}\,R^{\mathbf{i}}R^{\mathbf{i}} \tag{5.5}$$

in flat space, with a coordinate dependent potential

$$\mathcal{V}_{\mathbf{i}} = 2e^{2A_B}V_{\mathbf{i}}^{\rm tot}(Q_{\rm B}) + \frac{1}{2}A''_{\rm B}(\zeta) + \frac{1}{4}(A'_{\rm B}(\zeta))^2\,. \tag{5.6}$$

Restoring the fermionic part of (2.1), cf. [6], we arrive at the lagrangian

$$\mathcal{L}^{\mathbf{i}} = \frac{1}{2}\,\partial_\mu R^{\mathbf{i}}\partial^\mu R^{\mathbf{i}} + \frac{1}{2}\,\mathcal{V}_{\mathbf{i}}\,R^{\mathbf{i}}R^{\mathbf{i}} + \frac{1}{2}\,\overline{\chi}^{\mathbf{i}}\gamma^\mu\partial_\mu\chi^{\mathbf{i}} + \frac{1}{2}\,\mathcal{U}_{\mathbf{i}}\,\overline{\chi}^{\mathbf{i}}\chi^{\mathbf{i}}\,, \tag{5.7}$$

which is invariant under the global $N=1$ supersymmetry transformations

$$\delta R^{\mathbf{i}} = \overline{\chi}^{\mathbf{i}}\varepsilon\,, \qquad \delta\chi^{\mathbf{i}} = \gamma^\mu\varepsilon\,\partial_\mu R^{\mathbf{i}} + \mathcal{U}_{\mathbf{i}}\,R^{\mathbf{i}}\,\varepsilon\,, \qquad \gamma^\zeta\varepsilon = \varepsilon\,, \tag{5.8}$$

that explicitly descend from the Killing spinors of the domain wall solution (2.11). The fermionic mass term $\mathcal{U}_{\mathbf{i}}$ now serves as a prepotential for the scalar potential

$$\mathcal{V}_{\mathbf{i}} = \mathcal{U}'_{\mathbf{i}} + \mathcal{U}_{\mathbf{i}}^2 \tag{5.9}$$

and may be extracted from the corresponding mass term in the gauged supergravity [6], more specifically from expanding the so-called A_3 tensor around the background solution (2.11) to quadratic order of the inert scalars. The same prescription applies to all higher-dimensional supergravities. This shows that the existence of a prepotential is a direct consequence of the unbroken $N=1$ supersymmetry of the background.

[10] Usually the horospheric coordinate is called z, we call it ζ to distinguish it from the complex coordinate z we introduce later.

Now, considering the full lagrangian in the $\mathbf{9}$ sector, say, it is given by two copies of (5.7) with prepotentials $\mathcal{U}_{\mathbf{i}}$ related as $\mathcal{U}_{\mathbf{9}-} = -\mathcal{U}_{\mathbf{9}+}$. Closer inspection shows that on the total system we can realize another supersymmetry — i.e. in addition to (5.8) — with ε of opposite chirality $\gamma^\zeta\varepsilon = -\varepsilon$, and where the supermultiplets are $(R^{\mathbf{9}+}, \chi^{\mathbf{9}-})$, $(R^{\mathbf{9}-}, \chi^{\mathbf{9}+})$. In other words, the fact that not only $N = 1$ but $N = (1,1)$ supersymmetries are preserved implies that the potentials of the two scalars $R^{\mathbf{9}+}$, $R^{\mathbf{9}-}$ are superpartners in the sense of supersymmetric quantum mechanics, i.e. their prepotentials satisfy $\mathcal{U}_{\mathbf{9}-} = -\mathcal{U}_{\mathbf{9}+}$. This in turn leads to the correspondence between solutions of the equations of motion

$$R^- = (\partial_\zeta - \mathcal{U}_+)\, R^+\,, \tag{5.10}$$

which maps normalizable solutions into normalizable solutions.

Having understood where the prepotential structure comes from, let us see how it can be exploited. To facilitate later contact to standard 2d CFT expressions, we switch to a complex coordinate $z = \frac{1}{\sqrt{2}}\,(x^1 + ix^2)$ on the surfaces $\zeta =$ constant. Then the plane wave ansatz is, using a complex variable also for the 2-momenta $p = \frac{1}{\sqrt{2}}\,(p^1 + ip^2)$,

$$R^{\mathbf{i}}(\zeta, z) = e^{i(p\bar{z}+\bar{p}z)} R^{\mathbf{i}}(\zeta)\,, \tag{5.11}$$

and the fluctuation equations take the form

$$(-\partial_\zeta^2 + \mathcal{V}_{\mathbf{i}}\,)\, R^{\mathbf{i}} = -2|p|^2 R^{\mathbf{i}}\,, \tag{5.12}$$

with the coordinate ζ from (5.4) and effective potentials $\mathcal{V}_{\mathbf{i}}$ derived from superpotentials [5]

$$\begin{aligned}
\mathcal{U}_{\mathbf{1}} &= \frac{1}{32}\, e^A\,(y-1)(y+11)\,,\\
\mathcal{U}_{\mathbf{9}\pm} &= \mp\frac{1}{32}\, e^A\,(y-1)(y-5)\,,\\
\mathcal{U}_{\mathbf{4}\pm} &= \mp\frac{1}{32}\, e^A\,(y-1)(y+3)\,,
\end{aligned} \tag{5.13}$$

according to (5.9). Curiously, the scalar $\Phi^{\mathbf{9}+}$ admits an alternative superpotential $\tilde{\mathcal{U}}_{\mathbf{9}+} = -\mathcal{U}_{\mathbf{1}}$, a circumstance that is not explained by the above argument; carrying different $\mathrm{SO}(4)_{\mathrm{inv}}$ representations, $\Phi^{\mathbf{9}+}$ and $\Phi^{\mathbf{1}}$ can of course not be in the same $(1,1)$ supermultiplet nor be related by another bulk symmetry.

Now, factoring out the zero mode ($p = 0$) solution, and switching to a new variable s related to y by $y = \frac{5s^3-2}{2+s^3}$, the general solutions of the fluctuation equation (5.12) for $R^{\mathbf{9}+}$ and $R^{\mathbf{4}+}$ may be written as

$$\begin{aligned}
R^{\mathbf{9}+} &= (1+y)^{-1/4}\,\chi_{-2}\,,\\
R^{\mathbf{4}+} &= (1+y)^{1/12}(5-y)^{-1/3}\,\chi_0\,,
\end{aligned} \tag{5.14}$$

with functions $\chi_\alpha(s)$ satisfying

$$s\,\chi_\alpha'' + (1+\alpha)\,\chi_\alpha' - \frac{32\,|p|^2}{3}\,(2+s^3)\,\chi_\alpha = 0\,. \tag{5.15}$$

The remaining solutions are then automatically obtained by putting the supersymmetric quantum mechanics structure (5.10) to work:

$$\begin{aligned} R^{\mathbf{1}} &= (\partial_\zeta + \mathcal{U}_{\mathbf{1}})\, R^{\mathbf{9}+}\,, \\ R^{\mathbf{9}-} &= (\partial_\zeta + \mathcal{U}_{\mathbf{9}-})\, R^{\mathbf{9}+}\,, \\ R^{\mathbf{4}-} &= (\partial_\zeta + \mathcal{U}_{\mathbf{4}-})\, R^{\mathbf{4}+}\,. \end{aligned} \tag{5.16}$$

The two ordinary differential equations (5.15) for $\alpha = 0, -2$ thus comprise the entire dynamics of the inert scalar fluctuations. Remarkably, we will see below that the fluctuation equations for the active scalar and the metric reduce to the same universal equation (5.15)! Moreover, even the fluctuations in the vector sector eventually lead to equations of type (5.15) [5]. This differential equation is a special case of the *biconfluent Heun equation*, and we analyze it and its solutions in appendix B.

5.2 Distinguishing Δ_+ and Δ_-

The conformal dimension Δ of the operator dual to a scalar field of mass m in two dimensions is

$$\Delta = 1 \pm \sqrt{1+m^2}\,.$$

As first pointed out in [17], there is a certain mass range for scalars (here $-1 < m^2 \leq 0$) where the negative root can make physical sense, hence one scalar can correspond to operators of two different physical conformal dimensions $\Delta_\pm$. Here we summarize some recent arguments why the distinction between the two roots is obvious for an active scalar, but is not for two inert scalars belonging to conjugate roots $\Delta_\pm$ if they also have the same quantum numbers.[11]

To begin, one can view the inert scalar expansions (3.15) as being composed of two independent interlocking Taylor series, beginning at orders $\rho^{(d-\Delta)/2}$ and $\rho^{\Delta/2}$, respectively. The former can be thought of as the "source" (corresponding to standard AdS/CFT usage) series and the other the "response" series, in the language of [28]. In the series (3.15) for $\Delta = \Delta_+$, the exponent 1/4 is the $(d - \Delta_+)/2$, whereas the "response" Taylor series starts at order $\rho^{\Delta_+/2} = \rho^{3/4}$ and hence begins with the $\phi_{(1)}$ and $q_{(1)}$ terms. As we have seen these terms are, in fact, independent of the sources $\phi_{(0)}$ and $q_{(0)}$ in near-boundary analysis (perturbation theory around $\rho = 0$), but acquire a "nonperturbative" interrelation upon demanding regularity of the solution in the bulk interior.

For an operator with dimension given by the other root Δ_-, in our case $\Delta_- = 1/2$, the "response" series begins already at order $\Delta_-/2 = 1/4$, so essentially "source" and "response" are interchanged. Now, if one attempts to compute the 2-point function in the standard way, one is forced to add an additional quadratic counterterm, as explained in [28]. For a field with background (such as the active scalar q), this would generate terms linear in the source, which yields a nonvanishing 1-point function even when the source

[11] Despite appearances, the "±" notation for e.g. $\phi^{\mathbf{4}+}$ and $\phi^{\mathbf{4}-}$ is not intended to imply that $\phi^{\mathbf{4}+}$ has to be dual to an operator of dimension Δ_+; in fact, we will see that $\phi^{\mathbf{4}+}$ is dual to an operator of dimension Δ_-.

is set to zero. Hence, for the active scalar, selecting Δ_- is distinguishable from selecting Δ_+ in that the flows describe quite different physics, and the former choice presumably corresponds to turning on vevs in the boundary theory.[12]

For an inert scalar, however, these linear terms are not generated, and the only effect of the additional counterterm is that the coefficient of $\phi_{(1)}\phi_{(0)}$ in the action switches sign from $-(\Delta_+ - d/2)$ to $(\Delta_+ - d/2) = -(\Delta_- - d/2)$. Thus, the flow and fluctuation equations are capable of describing the correlation functions of operators of dimension Δ_- equally well as Δ_+. This means that it is not a priori obvious how to distinguish between inert scalars belonging to different roots $\Delta_\pm$ if they also have the same quantum numbers, in our case SO(4) quantum numbers. In the next section, we will construct a supersymmetry Ward identity for two-point functions including finite counterterms, which will, among other things, allow us to make this distinction in section 5.4.

The 2-point functions for operators of dimensions Δ_+ and Δ_- are related by a "massive" Legendre transformation, slightly generalized from [17] to include a $\phi^{\mathbf{i}}_{(0)}\phi^{\mathbf{i}}_{(0)}$ term in the action with a coordinate-independent coefficient $b_{\mathbf{i}}$. As we have seen, such terms arise as finite counterterms, but unlike the coefficient a for the active scalar, $b_{\mathbf{i}}$ is not simply fixed by the domain wall solution (cf. the discussion after (4.12)). In momentum space, the renormalized action for the inert scalar $\phi^{\mathbf{i}}$ can be written as

$$\hat{S}_{\text{ren}}(\phi^{\mathbf{i}}_{(0)}) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^2}\phi^{\mathbf{i}}_{(0)}(p)\phi^{\mathbf{i}}_{(0)}(-p)(f_+(|p|) + b_{\mathbf{i}}) ,$$

where f_+ would have been the two-point function of operators of dimension Δ_+, had we neglected the finite counterterm. The Legendre transform is effected by minimizing, just as in the massless case, the functional

$$J(\phi^{\mathbf{i}}_{(0)}, \phi^{\mathbf{i}}_{(1)}) = \hat{S}_{\text{ren}}(\phi^{\mathbf{i}}_{(0)}) - \int \frac{d^4p}{(2\pi)^2}\phi^{\mathbf{i}}_{(0)}(p)\phi^{\mathbf{i}}_{(1)}(-p)$$

with respect to $\phi^{\mathbf{i}}_{(0)}$, and in the linear approximation $\phi^{\mathbf{i}}_{(1)} = f_+(|p|)\phi^{\mathbf{i}}_{(0)}$. One solves for the extremum in $\phi^{\mathbf{i}}_{(0)}$ and substitutes in the original action to find

$$\hat{S}_{\text{ren}}(\phi^{\mathbf{i}}_{(1)}) = -\frac{1}{2}\int \frac{d^4p}{(2\pi)^2}\phi^{\mathbf{i}}_{(1)}(p)\phi^{\mathbf{i}}_{(1)}(-p)\frac{1}{f_+(|p|) + b_{\mathbf{i}}} .$$

In terms of the two-point function $f_-(|p|)$ of the Legendre-transformed ("response") theory, this action should have the same sign as $\hat{S}(\phi^{\mathbf{i}}_{(0)})$, in analogy to the sign of the kinetic energy in transforming from lagrangian to hamiltonian for a free massive particle. We can thus identify the two-point function of the operator with dimension Δ_- as

$$\langle \mathcal{O}^{\mathbf{i}}_\phi \mathcal{O}^{\mathbf{i}}_\phi \rangle = -\frac{1}{f_+(|p|) + b_{\mathbf{i}}} , \tag{5.17}$$

if $f_+(|p|)$ is the would-be two-point function for the operator with dimension Δ_+, when we neglect finite counterterms. This expression will be useful in section 5.4.

[12] So far, however, the machinery of holographic renormalization has only been applied to flows whose active scalar is associated with an operator of dimension $\Delta_+ > d/2$.

5.3 Counterterm fixed by Ward identity

Now, what is this constant $b_{\mathbf{i}}$ and how can it be computed? To answer this, we will derive the supersymmetry Ward identity between correlation functions of inert scalars, embodying the supersymmetric quantum mechanics structure. In this subsection, let us again suppress the representation index $\mathbf{i}$ for clarity and only restore it at the end. Consider in general the correlation functions of operators dual to a superpartner pair of scalar fields $\phi^{\pm}$. That is, the scalars $R^{\pm}$ defined by

$$\phi^{\pm} = C(\rho) R^{\pm} \tag{5.18}$$

for some function $C(\rho)$ (in our case $C(\rho) = e^{-A/2}$, but the explicit form is not important for the moment) satisfy the two fluctuation equations

$$(-\partial_\zeta^2 + \mathcal{V}^{\pm})\, R^{\pm} = -2|p|^2 R^{\pm}\,, \tag{5.19}$$

with $\mathcal{V}^{\pm} = \pm(\mathcal{U}^{\pm})' + (\mathcal{U}^{\pm})^2$. The two (normalizable) solutions of (5.19) are then related by (5.10). Let us first focus on R^+, say. Expanding R^+ in a series in ρ,

$$R^+ = R^+_{(0)} + \rho^{1/2} R^+_{(1)} + \rho R^+_{(2)} + \cdots\,, \tag{5.20}$$

the correlation functions of the dual operators in the complex coordinate $z = \frac{1}{\sqrt{2}}(x^1 + ix^2)$ will be obtained in the next section as (see [12])

$$\begin{aligned} \langle \mathcal{O}^+_\phi(z) \mathcal{O}^+_\phi(w) \rangle &= -\frac{1}{\sqrt{g_{(0)}(w)}} \frac{\delta \langle \mathcal{O}^+_\phi(z) \rangle}{\delta \phi_{(0)}(w)} \\ &= b + \frac{C_{(1)}}{C_{(0)}} + \frac{R^+_{(1)}(z)}{R^+_{(0)}(w)}\,. \end{aligned} \tag{5.21}$$

where $C = C_{(0)} + \sqrt{\rho}\, C_{(1)} + \cdots$ as usual. (Notice that in terms of the discussion of the previous section, this is for a "source" series; for a Δ_- operator, the correlator is actually the inverse of the right-hand side.) In the remaining part of this section we determine b.

From $d\zeta/d\rho = (-2\rho\, e^A)^{-1}$ it follows that

$$\frac{\partial}{\partial \zeta} = -2\rho^{1/2}(1 + \xi \rho^{1/2} + \cdots)\, \frac{\partial}{\partial \rho}\,, \tag{5.22}$$

where the constant ξ depends on the higher asymptotics in $A(\rho)$, and is $\xi = -1$ in our case, but the precise value turns out not to be important. With the expansion (5.20), we find

$$\begin{aligned} \partial_\zeta R^+ &= -R^+_{(1)} - \rho^{1/2}\,(2R^+_{(2)} + \xi R^+_{(1)}) + \cdots \\ \partial_\zeta^2 R^+ &= 2R^+_{(2)} + \xi R^+_{(1)} + \cdots \stackrel{!}{\equiv} R^+_{(0)}(\mathcal{V}^+_{(0)} + 2|p|^2) + \cdots\,. \end{aligned} \tag{5.23}$$

Here the last equality uses the fluctuation equation (5.19) and the expansion

$$\mathcal{V} = \mathcal{V}_{(0)} + \rho^{1/2}\mathcal{V}_{(1)} + \cdots \quad \Longrightarrow \quad \mathcal{V}^+_{(0)} = -\mathcal{U}^+_{(1)} + (\mathcal{U}^+_{(0)})^2\,. \tag{5.24}$$

inert scalar	$\mathcal{U}^{\mathbf{i}}_{(0)}$	$b_{\mathbf{i}} = \mathcal{U}^{\mathbf{i}}_{(0)} - 1/4$
9+	$1/8$	$-1/8$
9−	$-1/8$	$-3/8$
4+	$-1/8$	$-3/8$
4−	$1/8$	$-1/8$

Table 2: Finite counterterms for inert scalars.

The latter equation follows from $\mathcal{V} = \mathcal{U}' + \mathcal{U}^2$ and the chain rule (5.22). The expansion of the partner scalar R^- is then already determined from (5.10):

$$\begin{aligned} R^- &= (\partial_\zeta - \mathcal{U}^+)\, R^+ \\ &= -R^+_{(1)} - \rho^{1/2}\, R^+_{(0)}(\mathcal{V}^+_{(0)} + 2|p|^2) - (\mathcal{U}^+_{(0)} + \rho^{1/2}\mathcal{U}^+_{(1)})(R^+_{(0)} + \rho^{1/2} R^+_{(1)}) + \cdots \\ &= -(R^+_{(1)} + \mathcal{U}^+_{(0)} R^+_{(0)}) - \rho^{1/2} \left(\mathcal{U}^+_{(0)} R^+_{(1)} + (\mathcal{U}^+_{(0)})^2 R^+_{(0)} + 2|p|^2 R^+_{(0)}\right) + \cdots , \end{aligned}$$

from which we obtain the ratio

$$\frac{R^-_{(1)}}{R^-_{(0)}} = \mathcal{U}^+_{(0)} + 2\,|p|^2 \left(\mathcal{U}^+_{(0)} + \frac{R^+_{(1)}}{R^+_{(0)}}\right)^{-1} \tag{5.25}$$

and using $\mathcal{U}^+_{(0)} = -\mathcal{U}^-_{(0)}$, this gives rise to the supersymmetry Ward identity

$$\left(\frac{R^-_{(1)}}{R^-_{(0)}} + \mathcal{U}^-_{(0)}\right)\left(\frac{R^+_{(1)}}{R^+_{(0)}} + \mathcal{U}^+_{(0)}\right) = 2\,|p|^2 \tag{5.26}$$

for coefficients $R_{(1)}/R_{(0)}$ describing[13] two-point functions of two $N = (1,1)$ superpartner scalars. Comparing to (5.21), this determines the desired constant: $b^\pm$ is simply given by $\mathcal{U}^\pm_{(0)} - C_{(1)}/C_{(0)}$, i.e. $b^\pm$ is given by the boundary value of the prepotential, up to a constant shift due to the curved background. Using the expansion (3.14), one finds $C_{(1)}/C_{(0)} = 1/4$ in this flow. We summarize the results in table 2. These counterterm coefficients will be needed in the next section to show that the proper behavior of two-point functions emerges without dropping any terms by hand.

As one would expect, the argument leading to the supersymmetry Ward identity (5.26) can be seen in a different way on a two-dimensional surface ζ = constant, using known results in (1,1) superspace. Substituting $\partial_\zeta R^+ = \mathcal{U}^+ R^+ + R^-$ in (5.7), we find that for one chirality choice of the parameter ($\gamma^\zeta \epsilon = -\epsilon$), the supersymmetry transformation $\delta\chi_-$ has the superpartner R^- precisely in the place where the top component F of a (1,1) supermultiplet usually appears. In other words, the component transformations of a real scalar (1,1) superfield S with component fields $(R^+, \chi_+, \chi_-, R^-)$ precisely reproduce (5.8). Using standard results for relations between correlators of different components of the same superfield, one can finally recover a formula like (5.26).

[13] Again, for the Δ_- operator the correlator itself is actually the inverse of the expression in parenthesis, as we will see explicitly in the next section. The Ward identity (5.26) then takes the familiar schematic form $\langle \mathcal{O}\mathcal{O}\rangle = 2|p|^2 \langle \mathcal{O}\mathcal{O}\rangle$.

5.4 Two-point correlation functions

Since we study a flow to a fixed point, the boundary field theory is conformal in both IR and UV limits. This means we expect asymptotic power-law behavior of two-point functions (the only exception being the **4** marginal scalars in the IR, cf. table 1) on both sides. In other words, $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle \to p^{2\Delta-2}$ asymptotically, with two possibly different values $\Delta_{\rm UV}$ and $\Delta_{\rm IR}$. If we would apply the "leading nonanalytic term" (henceforth "old") prescription for AdS/CFT correlators, as successfully applied in e.g. [15], we would quickly be disappointed in this case. This is because the correlator we are interested in is asymptotically just a power of p, and in the "old" prescription it would have seemed that the correct power-law behavior would be indistinguishable among other monomial terms, coming from unphysical contact terms, that were summarily dropped.

In fact, in holographic renormalization, the requirement that only local counterterms may be added is sufficiently restrictive to single out exactly the right behavior. Indeed, $\sqrt{\gamma}\Phi\sqrt{\Box_\gamma}\Phi$ would yield a counterterm $\sim p$ but is nonlocal, whereas $\sqrt{\gamma}\Phi\Box_\gamma^n\Phi$ would yield a p^n counterterm but vanishes in the limit $\epsilon \to 0$ for all our $\Phi \sim \epsilon^{1/4}$ scalars. In other words, holographic renormalization does not allow us to "drop contact terms" as in the "old" prescription, which is fortunate since precisely some of those monomial terms are physical ones in our context. (Of course, there are many examples where that prescription is still applicable and useful.)

Now we compute the two-point functions of inert scalars. These two-point functions descend from uncoupled fluctuations that do not require an analysis of the kind we will perform for the active scalar and metric in the next section. They are obtained directly by taking the functional derivative of the one-point function (4.17), and are thus essentially (i.e. up to finite counterterms) encoded in the ratio $\phi^{\mathbf{i}}_{(1)}/\phi^{\mathbf{i}}_{(0)}$ in the expansion

$$\Phi^{\mathbf{i}} = \rho^{1/4}\left(\phi^{\mathbf{i}}_{(0)} + \sqrt{\rho}\,\phi^{\mathbf{i}}_{(1)} + \cdots\right), \tag{5.27}$$

of the solution to the fluctuation equations, after we impose regularity in the interior of the bulk. To compute this ratio, we recall that $\Phi^{\mathbf{i}}(\zeta, z) = e^{i(p\bar{z}+\bar{p}z)}\, e^{-A(\zeta)/2}\, R^{\mathbf{i}}(\zeta)$, and $R^{\mathbf{i}}(\zeta)$ is given explicitly in terms of biconfluent Heun functions in (5.14), (5.16). Using the following asymptotic expansions of the domain wall solution (2.9), (2.11)

$$\begin{aligned} e^{-A/2} &= \rho^{1/4} + \frac{1}{4}\rho^{3/4} + \frac{3}{64}\rho^{5/4} + \cdots, \\ y &= 1 + \rho^{1/2} + \frac{1}{4}\rho - \frac{5}{192}\rho^2 + \cdots, \\ s - s_0 &= \frac{1}{4}\rho^{1/2} + \frac{1}{16}\rho + \frac{1}{96}\rho^{3/2} + \cdots. \end{aligned} \tag{5.28}$$

(where we recall that s is the variable used in the fluctuation equation (5.15)), we eventually find

$$\frac{\phi^{\mathbf{1}}_{(1)}}{\phi^{\mathbf{1}}_{(0)}} = -\frac{1}{8} + \frac{8|p|^2}{\Psi_{-2}(p) - 2}$$

$$
\frac{\phi^{\mathbf{9}+}_{(1)}}{\phi^{\mathbf{9}+}_{(0)}} = \frac{1}{8} + \frac{\Psi_{-2}(p)}{4}\,, \qquad \frac{\phi^{\mathbf{9}-}_{(1)}}{\phi^{\mathbf{9}-}_{(0)}} = \frac{3}{8} + \frac{8|p|^2}{\Psi_{-2}(p)}
$$
$$
\frac{\phi^{\mathbf{4}+}_{(1)}}{\phi^{\mathbf{4}+}_{(0)}} = \frac{3}{8} + \frac{\Psi_{0}(p)}{4}\,, \qquad \frac{\phi^{\mathbf{4}-}_{(1)}}{\phi^{\mathbf{4}-}_{(0)}} = \frac{1}{8} + \frac{8|p|^2}{\Psi_{0}(p)} \tag{5.29}
$$

with the ratios Ψ_α from the expansion of the regular Heun function in (B.2). Consulting table 2, we see that the effect of the $Q^2\Phi^2$ counterterms with the coefficients $b_{\mathbf{i}}$ determined above is precisely to cancel the additive constants in these expressions. The resulting correlators may be expressed in terms of the Heun function coefficients $\Psi_\alpha(p)$ (discussed in appendix B):

$$
\begin{aligned}
\langle \mathcal{O}^{\mathbf{1}}_\phi(-p)\mathcal{O}^{\mathbf{1}}_\phi(p)\rangle &= k\,\frac{\Psi_{-2}(p)-2}{8\,(2+8|p|^2-\Psi_{-2}(p))}\,, \\
\langle \mathcal{O}^{\mathbf{9}+}_\phi(-p)\mathcal{O}^{\mathbf{9}+}_\phi(p)\rangle &= \frac{k}{2\Psi_{-2}(p)}\,, \qquad \langle \mathcal{O}^{\mathbf{9}-}_\phi(-p)\mathcal{O}^{\mathbf{9}-}_\phi(p)\rangle = \frac{k|p|^2}{\Psi_{-2}(p)}\,, \\
\langle \mathcal{O}^{\mathbf{4}+}_\phi(-p)\mathcal{O}^{\mathbf{4}+}_\phi(p)\rangle &= \frac{k}{2\Psi_{0}(p)}\,, \qquad \langle \mathcal{O}^{\mathbf{4}-}_\phi(-p)\mathcal{O}^{\mathbf{4}-}_\phi(p)\rangle = \frac{k|p|^2}{\Psi_{0}(p)}\,,
\end{aligned} \tag{5.30}
$$

and all mixed two-point functions vanish. Here we have restored the factor $k/8$ which comes from properly normalizing the supergravity action in (4.13). This k is the level of the current algebra in the ultraviolet CFT, and it depends on the number N of D5-branes as $k \sim N^2$ [7]. We only restore this factor in final results for correlators.

The Heun function coefficients $\Psi_\alpha(p)$ are plotted in figures 3 and 4 in appendix B; analytic expressions for the large and small p asymptotics are obtained in (B.10)–(B.12), from which one may immediately derive the UV and IR asymptotics of these correlators. Appealing to (5.17), we had to invert some of the ratios (5.29), namely those corresponding to the operators of UV conformal dimension $\Delta_- = 1/2$. In the **4** and the **9** sector, we want to make the distinction between Δ_+ and Δ_-. As it turns out, this distinction is unique for the scalars in the **4**: it is only upon inverting the shifted ratio $(\phi^{\mathbf{4}+}_{(1)}/\phi^{\mathbf{4}+}_{(0)} - 3/8)$ that the resulting correlator has the correct logarithmic behavior from (B.11), as expected for a $\Delta_{\mathrm{IR}} = 1$ operator. Tracing this identification back through the flow, we conclude that in the UV it is $\Phi^{\mathbf{4}+}$ which is associated with $\Delta_- = 1/2$, whereas $\Phi^{\mathbf{4}+}$ corresponds to $\Delta_+ = 3/2$. Thus, the computation of correlators throughout the flow allows us to make this distinction which would have been impossible to derive from a near boundary analysis around the UV boundary.

In the **9** sector, on the other hand, the ambiguity is left unresolved. We have chosen to associate $\Phi^{\mathbf{9}+}$ with $\Delta_- = 1/2$ and $\Phi^{\mathbf{9}-}$ with $\Delta_+ = 3/2$ but could also have done it the other way round, both choices are compatible with the correct asymptotics of $\Delta_- = 1/2$, $\Delta_+ = 3/2$ operators in the UV and IR.

For the singlet $\Phi^{\mathbf{1}}$, we know that it corresponds to a $\Delta_- = 1/2$ operator in the UV, since it is the superpartner of the active scalar which has dimension $\Delta_+ = 3/2$ (see discussion in section 5.2). For this inert singlet scalar, we have fixed the coefficient of the $Q^2\Phi^2$ counterterm such that the resulting correlator in (5.30) satisfies the supersymmetry

Ward identity with the correlation function derived for the active scalar in (6.20) below. (It is quite a nontrivial consistency check that this Ward identity may indeed be satisfied by just adding the proper constant to the ratio in (5.29).) This correlator leaves a minor puzzle in the IR: $\Psi_{-2}(p)$ goes linear in p for small p, so the correlation function goes to a constant rather than like the $|p|^3$ which one would have expected for $\Delta_{\rm IR} = 5/2$. However, irrelevant operators decouple in the infrared, so it is difficult to know whether the method works straightforwardly for operators crossing over from relevant to irrelevant along the flow; certainly UV-irrelevant operators cannot be treated the same way as UV-relevant operators [10].

6. Two-point functions of active scalar and stress-energy tensor

6.1 Fluctuation equations

Unlike the inert scalars, the active scalar fluctuation couples to the metric fluctuation at linear order in the equation of motion. This was a tough obstacle (see e.g. [16]) until resolved in [11] (with earlier progress in [29]). The resolution involved working with "gauge invariant quantities", therefore we vow to only work with such quantities, in a sense to be made precise below.

To compute the quadratic fluctuations of the active scalar and the metric, the inert scalars may be switched off as they do not contribute to these couplings. We use complex coordinates $z = \frac{1}{\sqrt{2}}(x^1 + ix^2)$ on the boundary, and parametrize the domain wall background and the fluctuations as

$$\begin{aligned} ds^2 &= e^{2A(r)}\Big(2(1+h+|p|^2 H)\,dz\,d\bar z + \bar p^2(H+H_\perp)\,dz\,dz + p^2(H-H_\perp)\,d\bar z\,d\bar z\Big) + \\ &\quad + (1+h_{rr})\,dr^2 \\ q &= q_{\rm B} + \varphi \end{aligned} \tag{6.1}$$

with the same plane wave ansatz in the complex z coordinate as was used earlier:

$$\varphi(z,\bar z,r) = e^{i(p\bar z+\bar p z)}\,\varphi(r)\,, \qquad \text{etc.} \tag{6.2}$$

for all the fluctuations. Linearizing the equations of motion around an arbitrary domain wall (2.9), we obtain

$$\begin{aligned} H_\perp' &= 0\,, \\ H'' - 4WH' &= -e^{-2A}\,h_{rr}\,, \\ 2h_{rr}\,W + h' &= -4\varphi W'\,, \\ |p|^2 e^{-2A}\,h &= 2|p|^2\,WH' + (2\varphi' - h_{rr}W' - 2\varphi W'')\,W'\,, \end{aligned} \tag{6.3}$$

in terms of the superpotential $W(Q)$. In the above equations, primes on the superpotential W denote derivatives with respect to Q, whereas all other primes refer to derivatives with respect to the radial variable r. The first equation in (6.3) is a manifestation of the well-known fact that in three spacetime dimensions there are no transverse-traceless degrees of freedom in the metric. As explained in [11], the ansatz (6.1) does not completely fix the bulk diffeomorphisms, but leaves the freedom of "gauge transformations" generated by vector fields $(\xi^z, \xi^{\bar{z}}, \xi^r)$ satisfying

$$\partial_r \xi^z = -p e^{-2A}\, \xi^r\,, \qquad \partial_r \xi^{\bar{z}} = -\bar{p} e^{-2A}\, \xi^r\,, \tag{6.4}$$

under which the fluctuations transform as

$$\begin{aligned} \delta\, h &= 2\xi^r A'\,, & \delta\, h_{rr} &= 2\partial_r \xi^r\,, \qquad \delta\, \varphi = \xi^r q_{\mathrm{B}}'\,, \\ \delta\, H &= \frac{1}{|p|^2}\,(\bar{p}\xi^z + p\xi^{\bar{z}})\,, & \delta\, H' &= -2e^{-2A}\xi^r\,, \\ \delta\, H_\perp &= -\frac{1}{|p|^2}\,(\bar{p}\xi^z - p\xi^{\bar{z}})\,, & \delta\, H_\perp' &= 0\,. \end{aligned} \tag{6.5}$$

It therefore seems appropriate to cast the above fluctuation equations in equations for the "gauge invariant" objects

$$\begin{aligned} \mathcal{J}_1 &= h + 4\frac{W}{W'}\,\varphi\,, \qquad \mathcal{J}_2 = -2H' - \frac{4e^{-2A}}{W'}\,\varphi\,, \\ \mathcal{R} &= h_{rr} - \frac{2(\varphi' - \varphi W'')}{W'}\,. \end{aligned} \tag{6.6}$$

This leads to

$$\begin{aligned} |p|^2\, e^{-2A} \mathcal{J}_1 &= W\mathcal{R}' - \big(4W^2 + (W')^2 - 2WW''\big)\,\mathcal{R}\,, \\ |p|^2\, \mathcal{J}_2 &= (-2W'' + 4W)\,\mathcal{R} - \mathcal{R}'\,, \\ \mathcal{J}_1' &= -2\,W\mathcal{R}\,, \end{aligned} \tag{6.7}$$

which we may combine into the second order equation for $\mathcal{R}$

$$0 = \mathcal{R}'' + (2W'' - 8W)\,\mathcal{R}' + \big(2|p|^2 e^{-2A} + 16W^2 - 4(W')^2 - 8WW'' + 2W'W'''\big)\,\mathcal{R}\,. \tag{6.8}$$

The analogous equation in five dimensions was obtained in [11]. Specializing this equation to the superpotential W from (2.8) and transforming to $y = \cosh(\sqrt{2}Q)$, we find

$$\begin{aligned} 2|p|^2\, e^{-2A}\mathcal{R}(y) =\; & \frac{1}{64}(y-5)^2(y-1)^2(y+1)^2\mathcal{R}''(y) + \\ & + \frac{1}{64}(y-5)(y-1)(y+1)(11+5y^2)\mathcal{R}'(y) + \\ & + \frac{1}{64}\big(35 + y(80 + 10y + 3y^3)\big)\,\mathcal{R}(y)\,. \end{aligned} \tag{6.9}$$

Dividing out its zero mode as in (5.14),

$$\mathcal{R} = (y-1)(y-5)^{-\frac{4}{3}}(y+1)^{-\frac{2}{3}}\,\tilde{\chi}\,, \tag{6.10}$$

and performing the same change of variables $y = \frac{5s^3-2}{2+s^3}$ as that leading to (5.15), eq. (6.9) reduces to

$$s\,\tilde{\chi}'' - \frac{4+5s^3}{2+s^3}\,\tilde{\chi}' - \frac{32\,|p|^2}{3}\,(2+s^3)\,\tilde{\chi} = 0\,. \tag{6.11}$$

We now proceed to show that also this equation may be reduced to one of the two Heun equations (5.15). Indeed, let χ be the regular solution of (5.15) with $\alpha = -2$, then

$$\tilde{\chi} = \chi - \frac{1}{2} s\chi' \tag{6.12}$$

is a solution of (6.11), i.e. we can extract all the asymptotics data from our previously obtained solution! In particular, after some computation it is seen that the solution of (6.8) regular in the bulk interior has the expansion

$$\mathcal{R} = \text{const} \times \left(\sqrt{\rho} + \frac{-2 + 32|p|^2 + \Psi_{-2}(p)}{4(-2 + \Psi_{-2}(p))} \rho + \cdots \right), \tag{6.13}$$

with $\Psi_{-2}(p)$ from (B.2). Using this expansion we can now procced to compute two-point functions.

6.2 Two-point correlation functions

The rest of the computation is straightforward. Linearizing the one-point function of the stress-energy tensor (4.22) around the background using

$$g_{ij} = \rho e^{2A_{\mathrm{B}}} (\eta_{ij} + h_{ij}) \,, \tag{6.14}$$

leads to

$$\langle T_{ij} \rangle = \eta_{ij} \left(\frac{1}{4} \operatorname{Tr} h_{(1)} - \frac{1}{2} \operatorname{Tr} h_{(2)} + \frac{7}{16} \varphi_{(0)} - \frac{1}{2} \varphi_{(1)} \right) - \frac{1}{4} h_{(1)ij} + \frac{1}{2} h_{(2)ij} \,, \tag{6.15}$$

or, in the complex notation from (6.1),

$$\begin{aligned} \langle T_{z\bar{z}} \rangle &= \frac{1}{4}(h_{(1)} + |p|^2 H_{(1)}) - \frac{1}{2}(h_{(2)} + |p|^2 H_{(2)}) + \frac{7}{16} \varphi_{(0)} - \frac{1}{2} \varphi_{(1)} \,, \\ \langle T_{zz} \rangle &= \bar{p}^2 \left(-\frac{1}{4} H_{(1)} + \frac{1}{2} H_{(2)} \right) , \\ \langle T_{\bar{z}\bar{z}} \rangle &= p^2 \left(-\frac{1}{4} H_{(1)} + \frac{1}{2} H_{(2)} \right) . \end{aligned} \tag{6.16}$$

where the subscripts in parentheses denote coefficients in the ρ expansion, $H = H_{(0)} + \sqrt{\rho} H_{(1)} + \rho H_{(2)} + \cdots$ as usual, and we have used that $H_\perp = \text{constant}$. Expanding the gauge invariant quantities (6.6) yields

$$\begin{aligned} \mathcal{R} &\equiv \mathcal{R}_{(0)} \sqrt{\rho} + \mathcal{R}_{(1)} \rho + \cdots \\ &= \left(\frac{1}{2} \varphi_{(0)} - 4\varphi_{(1)} \right) \sqrt{\rho} + \left(-\frac{15}{16} \varphi_{(0)} + \varphi_{(1)} - 8\varphi_{(2)} \right) \rho + \cdots \,, \\ \mathcal{J}_{(1)} &= (h_{(0)} + 4\varphi_{(0)}) + \left(h_{(1)} + \frac{3}{2} \varphi_{(0)} + 4\varphi_{(1)} \right) \sqrt{\rho} + \\ &\quad + \left(h_{(2)} + \frac{15}{32} \varphi_{(0)} + \frac{3}{2} \varphi_{(1)} + 4\varphi_{(2)} \right) \rho + \cdots \,, \\ \mathcal{J}_{(2)} &= 2H_{(1)} \sqrt{\rho} + \left(4H_{(2)} + 8\varphi_{(0)} \right) \rho + \cdots \,. \end{aligned} \tag{6.17}$$

One sees that the coefficient $\mathcal{R}_{(0)} = \frac{1}{2}\varphi_{(0)} - 4\varphi_{(1)}$ is proportional to the one-point function $\langle O_q \rangle$ of the active scalar (just set $\phi^{\mathbf{i}} = 0$ in (4.21)), hence that the one-point function is gauge invariant as one would have hoped. From the equations of motion (6.7), we then find that we can express all the perturbative coefficients in terms of $\varphi_{(0)}$, $h_{(0)}$, and the ratio $\mathcal{R}_{(1)}/\mathcal{R}_{(0)}$ which has been determined in (6.13) above:

$$\begin{aligned}
H_{(1)} &= 0\,, & H_{(2)} &= \frac{1}{2}\,h_{(0)} + \frac{h_{(0)} + 4\varphi_{(0)}}{4(\mathcal{R}_{(1)}/\mathcal{R}_{(0)}) - 5}\,, \\
h_{(1)} &= -2\,\varphi_{(0)}\,, & h_{(2)} &= -\frac{1}{4}\,\varphi_{(0)} + \frac{2|p|^2(h_{(0)} + 4\varphi_{(0)})}{4(\mathcal{R}_{(1)}/\mathcal{R}_{(0)}) - 5}\,, \\
\varphi_{(1)} &= \frac{1}{8}\,\varphi_{(0)} - \frac{2|p|^2(h_{(0)} + 4\varphi_{(0)})}{4(\mathcal{R}_{(1)}/\mathcal{R}_{(0)}) - 5}\,.
\end{aligned} \tag{6.18}$$

Substituting this into (4.21), (6.15), we find

$$\begin{aligned}
\langle \mathcal{O}_q \rangle &= \frac{2|p|^2\,(h_{(0)} + 4\varphi_{(0)})}{4(\mathcal{R}_{(1)}/\mathcal{R}_{(0)}) - 5}\,, \\
-\frac{1}{|p|^2}\,\langle T_{z\bar{z}} \rangle = \frac{1}{\bar{p}^2}\,\langle T_{zz} \rangle = \frac{1}{p^2}\,\langle T_{\bar{z}\bar{z}} \rangle &= \frac{1}{4}\,h_{(0)} + \frac{h_{(0)} + 4\varphi_{(0)}}{8(\mathcal{R}_{(1)}/\mathcal{R}_{(0)}) - 10}\,,
\end{aligned} \tag{6.19}$$

which gives the two-point correlation functions

$$\begin{aligned}
\langle \mathcal{O}_q \mathcal{O}_q \rangle &= \frac{k|p|^2}{4(\mathcal{R}_{(1)}/\mathcal{R}_{(0)}) - 5} = \frac{k|p|^2\,(-2 + \Psi_{-2}(p))}{4(2 + 8|p|^2 - \Psi_{-2}(p))}\,, \\
\langle \mathcal{O}_q T_{z\bar{z}} \rangle &= \frac{-k|p|^2}{16\mathcal{R}_{(1)}/\mathcal{R}_{(0)} - 20} = -k|p|^2\,\frac{-1 + \frac{1}{2}\Psi_{-2}(p)}{8(2 + 8|p|^2 - \Psi_{-2}(p))}\,, \\
\langle T_{z\bar{z}} T_{z\bar{z}} \rangle &= -\frac{k}{32}\,|p|^2\,\frac{4\mathcal{R}_{(1)}/\mathcal{R}_{(0)} - 3}{4\mathcal{R}_{(1)}/\mathcal{R}_{(0)} - 5} = -\frac{k|p|^4}{8}\left(\frac{1}{8|p|^2} + \frac{1}{2 + 8|p|^2 - \Psi_{-2}(p)}\right),
\end{aligned} \tag{6.20}$$

where we have substituted the expansion (6.13) for $\mathcal{R}_{(1)}/\mathcal{R}_{(0)}$. The first two correlators are, now that the smoke has cleared, trivially related by[14]

$$T_{z\bar{z}} = \beta\mathcal{O}_q + \mathcal{A}\,, \tag{6.21}$$

where $\beta = -1/2$ is the classical β function due to the classical scaling of the coefficient $q_{(0)}$, when this coefficient is viewed as the holographic coupling constant in the deformation $\mathcal{L}_{\mathrm{CFT}} + q_{(0)}\mathcal{O}_q$. As in [11], the fact that the β function is classical could be ascribed to a nonrenormalization theorem to the effect that the only contribution to the β function could come from an anomalous dimension of the operator it multiplies, and if this operator is protected, there are no quantum corrections to scaling. After all, the deformation preserves some supersymmetry, so this is perhaps not so surprising.

[14] See e.g. [13, eq. (6.13)], and use $\langle \mathcal{A}\mathcal{O}_q \rangle_{\mathrm{B}} = 0$.

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It is straightforward to see that the two-point correlator of the active scalar has the correct (linear $|p|$) UV behavior, corresponding to an operator of conformal dimension 3/2. Moreover, comparing to (5.30), we see that it indeed satisfies the supersymmetry Ward identity $\langle \mathcal{O}_q \mathcal{O}_q \rangle = 2|p|^2 \langle \mathcal{O}^{\mathbf{1}}_\phi \mathcal{O}^{\mathbf{1}}_\phi \rangle$. The third correlator, $\langle T_{z\bar{z}} T_{z\bar{z}} \rangle$, and its asymptotics will be studied in more detail in the next section, since it is related to the C function along the renormalization group flow.

7. The C function

Given the stress-energy 2-point functions, we may follow Zamolodchikov's original construction [18] to compute the C *function*, a function on the space of couplings that is monotonic along the flow and interpolates between the central charges of the conformal fixed points. The variety of possible tensor structures in $d > 2$ makes this construction difficult to generalize to higher dimensions [30, 31]; in particular, the straightforward proof of monotonicity as a consequence of unitarity is tailored to fit the two-dimensional case. Nevertheless, there have been several proposals for defining monotonic C functions by holography [32, 19, 2]. In particular the "holographic C function" of [19, 2] is a very simple proposal in terms of the supergravity superpotential W, which reads $C_{\text{hol}} \equiv -1/W(Q_{\text{B}})$ in two dimensions. It is monotonic as a function of the bulk radial coordinate, assuming a fairly weak positive-energy condition in the bulk supergravity. Applied to our superpotential (2.8), we find

$$C_{\text{hol}} = -\frac{3k}{W(Q_{\text{B}})} = 3k \left(1 - \frac{1}{2}\sqrt{\rho} + \frac{3}{16}\rho + \cdots \right). \tag{7.1}$$

with normalization adapted to our conventions. Monotonicity may be verified directly.

We will now compute Zamolodchikov's C function from the holographic $\langle TT \rangle$ correlators obtained above. As a first check of the $\langle TT \rangle$ correlators (6.20), we expand their asymptotic behavior using (B.10), (B.12), and find

$$\begin{aligned} \langle T_{z\bar{z}}(p) T_{z\bar{z}}(-p) \rangle &\overset{\text{UV}}{=} -\frac{k}{32}\left(|p|^2 - \frac{1}{2\sqrt{2}}|p| + \frac{5}{32} + \cdots \right), \\ \langle T_{z\bar{z}}(p) T_{z\bar{z}}(-p) \rangle &\overset{\text{IR}}{=} -\frac{k}{32}\left(\frac{1}{2}|p|^2 + 2|p|^4 - \frac{4\sqrt{2}}{\sqrt{3}}|p|^5 + \cdots \right). \end{aligned} \tag{7.2}$$

In particular, this shows that in fact $c_{\text{IR}}/c_{\text{UV}} = 1/2$, which is precisely what is expected (see eq. (2.5)). To construct Zamolodchikov's C function, we first Fourier transform the $\langle T_{z\bar{z}} T_{z\bar{z}} \rangle$ correlator back to the complex z plane as

$$\begin{aligned} \langle T_{z\bar{z}}(z) T_{z\bar{z}}(0) \rangle &= \frac{1}{2\pi} \int dp\, d\bar{p}\, e^{i(p\bar{z}+\bar{p}z)} \, \langle T_{z\bar{z}}(p) T_{z\bar{z}}(-p) \rangle \\ &=: \Box\Box\, \Omega(t) = \frac{4}{|z|^4}\left(\Omega'' - \Omega''' + \frac{1}{4}\Omega'''' \right), \end{aligned} \tag{7.3}$$

defining the function $\Omega(t)$ with $t = \frac{1}{2}\log(\mu^2 |z|^2)$, that is, $\partial_t = |z| \partial_{|z|}$, and primes denote derivatives with respect to t. This function encodes Zamolodchikov's C function [18] as

$$C_{\text{Zam}} = -96 \left(\Omega' - \Omega'' + \frac{1}{4}\Omega''' \right), \tag{7.4}$$

such that $C' = -24\,|z|^4\, \langle T_{z\bar{z}}(x)T_{z\bar{z}}(0)\rangle$. Specifically we find the integral representations

$$\begin{aligned}
\Omega(t) &= \int d|p|\,|p|^{-3}\,J_0(2|pz|)\,\langle T_{z\bar{z}}(p)T_{z\bar{z}}(-p)\rangle\,, \\
\Omega'(t) &= -\int d|p|\,2|p|^{-2}\,J_1(2|pz|)\,\langle T_{z\bar{z}}(p)T_{z\bar{z}}(-p)\rangle\,, \\
C_{\text{Zam}} &= 192\int d|p|\Big\{|p|^{-2}|z|\,\big(|pz|J_0(2|pz|)+(|pz|^2-1)J_1(2|pz|)\big)\times \\
&\qquad\qquad \times\langle T_{z\bar{z}}(p)T_{z\bar{z}}(-p)\rangle\Big\}\,.
\end{aligned} \tag{7.5}$$

With proper regularization, the last integral gives

$$\int d|p|\Big\{|p|^{-2}|z|\Big(|pz|J_0(2|pz|)+(|pz|^2-1)J_1(2|pz|)\Big)\,|p|^n\Big\} = \frac{\pi\,n\,|z|^{2-n}}{4\sin(\frac{n\pi}{2})\Gamma(2-\frac{n}{2})\Gamma(-\frac{n}{2})}\,, \tag{7.6}$$

for the polynomial terms in $\langle T_{z\bar{z}}T_{z\bar{z}}\rangle$. From the exact asymptotics (7.2), we may then derive the small distance behavior of the C function

$$C_{\text{Zam}} = 3k\left(1-\frac{1}{4\sqrt{2}}\,|z| + \mathcal{O}(|z|^3)\right). \tag{7.7}$$

For the full C function we have to insert in (7.5) the complete expression from (6.20), with Ψ_{-2} from appendix B. The numerical result is plotted in figure 2, and it is another consistency check of the correlators (6.20) that the function comes out strictly monotonic, i.e. the holographic correlators indeed reproduce the properties of a unitary field theory.

To compare the result to the proposal (7.1) above, we recall that in holographic flows the RG scale μ is introduced by the AdS isometry

$$\rho \to \mu^2\rho\,, \qquad x^i \;\to\; \mu x^i\,, \tag{7.8}$$

connecting the energy scale to the bulk radial coordinate. At fixed ρ this converts the superpotential $W(Q_{\text{B}}(\rho=\mu^2\rho_0))$ into a function of μ. Likewise, C_{Zam} turns into a function of μ upon relating the boundary radius $|z|$ to a fixed value $|z| = \mu|z_0|$. Normalizing C_{hol} and C_{Zam} to unity in the UV and fixing the values of $|z_0|$ and ρ_0 such that the first derivatives C'_{hol} and C'_{Zam} coincide in the UV, we plot the two functions in figure 2.

Both functions are strictly monotonic with surprisingly similar shapes. We should stress that the discrepancy in figure 2 does not mean that the different proposals are incompatible — the computation of the C function is in any case scheme dependent — but may rather indicate that the identification of the boundary energy scale with the radial AdS variable (7.8) requires corrections away from criticality.

A comparison of the different C function proposals in the five-dimensional confining flow of [4] was done in [33], see also [34]. Some general considerations in higher-dimensional conformal-to-conformal flows are presented in [35]. Note that in higher dimensions the relevant central function descends from the two-point correlators of the transverse-traceless part of the stress-energy tensor, which drastically simplifies the computation. In two dimensions one instead has to go through the full procedure of decoupling the active scalar and metric fluctuations, leading to the fluctuation equation (6.8) and its solution presented in the previous section.

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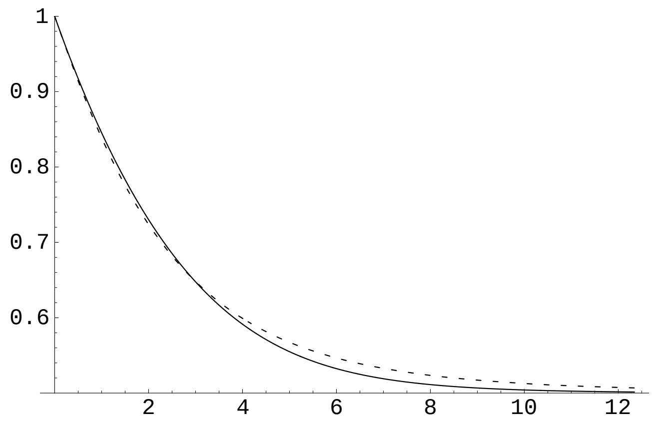

Figure 2: Holographic C function (7.1) (solid) and Zamolodchikov's C function (7.4) (dashed) from holographic correlators as functions of μ after normalizing and rescaling such as to match the first derivative in the UV.

A challenging goal would of course be the computation of the $\langle TT\rangle$ correlators directly in the CFT as input to C_{Zam}. Comparison to the correlators obtained by the holographic methods presented here could provide a demanding test of the correspondence.

8. Outlook

In this paper, we have computed correlators along a renormalization group flow interpolating between conformal field theories. As a main result, we have obtained the two-point correlation functions $\langle TT\rangle$, $\langle T\mathcal{O}\rangle$, and $\langle \mathcal{O}\mathcal{O}\rangle$ of the stress-energy tensor, and operators $\mathcal{O}$ dual to supergravity scalars, both active and inert. We used the $\langle TT\rangle$ correlators to compute the Zamolodchikov C function and compare it to the holographic proposal for the C function in terms of the supergravity superpotential. Since we summarized the results in the introduction, here we only give some brief remarks on future directions.

First, one could study correlators of currents J^i in the boundary theory. As in [12], these currents correspond to symmetries broken by the deformation $q_{(0)}\mathcal{O}_q$ (here SO(4)), although their analysis was facilitated by the remaining $\mathrm{U}(1)_R$ symmetry of the IR theory, whereas our IR theory has no R-symmetry at all. Presumably this analysis is nevertheless straightforward on the supergravity side, given the fact that also the vector fluctuation equations reduce to the same biconfluent Heun equation [5].

More interestingly, it would be worth to study the CFT side in more detail than has been done here. In fact, it should be possible to compute correlators such as $\langle TT\rangle$ given the deformation $\mathcal{L}_{\mathrm{CFT}} + q_{(0)}\mathcal{O}_q$, where $q_{(0)}$ is the holographic coupling. One could hope to show that this deformation is integrable, and to compute correlators exactly.

One can also proceed to apply these methods in other settings. Perhaps the example of greatest direct physical interest would be domain wall solutions in five-dimensional gauged supergravity. Even though the 5d flow equations in [2] could not be solved exactly, one could see how far one can take a numerical analysis of correlators there. Unfortunately, as emphasized in section 2, it will probably be well-nigh impossible to achieve any confidence in

old	q	q	$\phi^{\mathbf{i}}$	φ	(x_i, y_i)	g^2V	$V_{\mathbf{i}}$	κ^2
new	$\sqrt{2}Q$	q	$\Phi^{\mathbf{i}}/\sqrt{2}$	$\phi^{\mathbf{i}}$	(Z_1, Z_2)	V	$V_{\mathbf{i}}^{\text{tot}}$	κ

Table 3: Notation: reference [5] vs. this paper.

the numerics if one cannot find any analytic help. On the other hand, it might be possible to construct exactly soluble setups in other five-dimensional cases than that of [2]. Encouraged by the solution in section 2, and recalling that it was crucial for the explicit construction of [5] to set the two D5-brane charges equal (other values of this ratio would have resulted in a vertical stretching of the potential in figure 1 such that the flow trajectory would no longer be a straight line), one might look for simplifying special values of coupling constants in various five-dimensional gauged supergravities. In an analogous half-maximal theory in 5d, in which first-order equations for flows were studied in [36], there is an $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ non-abelian gauge symmetry but apparently no free parameter of this kind.

As a matter of principle, it would be interesting to compute n-point functions for $n > 2$. Although the full Dirichlet problem may be practically unsoluble, one can imagine solving nonlinear fluctuation equations in perturbation theory. Recently, an outline of an example for $n = 4$ was given in [13, section 5.9].

Finally, one could compute correlators in other RG flows in two-dimensional conformal field theories, further pursuing the idea of three-dimensional gauged supergravity as a tool in this field. Of particular interest and in principle accessible with our tools are for example flows to nonsupersymmetric but stable fixpoints, the analysis of marginal deformations of the CFT describing the D1-D5 system (cf. [37, 38, 39]), and flows in the maximally supersymmetric theory of [8] and their role in a supergravity description of matrix string theory (cf. [40]).

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A. Notation

Since we made many small notational changes with respect to [5], where the domain wall solution was constructed and discussed, it might be useful to collect the differences. The reason for these differences is the discrepancy between the 3d supergravity literature and holography literature (such as [11, 12]) and in this paper we stick close to the latter. The differences between *old* (reference [5]) and *new* (this paper) notation are summarized in table 3.

In addition, the two integration constants in the domain wall solution (2.11) are rescaled relative to [5], so that also the expansions (3.14) appear different; in both cases the integration constants can be fixed by demanding $q_{(0)\mathrm{B}} = 1$ and $g_{(0)\mathrm{B}} = 1$ in the domain wall solution. Finally, the signature in [5] is lorentzian $(+ - -)$ except for in section 4 where it is Wick rotated to riemannian signature, and here it is riemannian throughout.

B. Analytics of the fluctuation equations

We have seen in the main text that the entire set of fluctuation equations around the background (2.11) may be reduced to equations of the type

$$s\chi_\alpha'' + (1+\alpha)\,\chi_\alpha' - P^2\,(2+s^3)\chi_\alpha = 0\,, \tag{B.1}$$

for $\alpha = 0, -2$, and with $P = \sqrt{32/3}\,|p|$. This appendix is devoted to a closer study of this differential equation and the properties of its solutions. Let us recall that along the flow the variable s runs from $s = 1$ at the AdS boundary (the UV) to $s = \infty$ in the AdS interior. This implies that the two-point correlation functions are encoded in the first coefficient of the expansion at $s = 1$

$$\chi_\alpha(s) = \chi_\alpha(1)\,(1 + (s-1)\,\Psi_\alpha + \cdots)\,, \tag{B.2}$$

of the solution χ_α regular in the interior $s \to \infty$. The coefficient Ψ_α is uniquely defined as a function of the parameter P by the requirement that χ_α is regular as $s \to \infty$. This procedure is entirely analogous to that used in the analysis of the five-dimensional flows [11, 28], where the corresponding differential equations may be reduced to hypergeometric equations. In those cases, however, the regularity condition was imposed at a curvature singularity.

A first inspection shows that (B.1) has two singular points (zero and infinity) with "s-rank $\{1;3\}$", in the language of [41]. This means that it may be obtained from a Fuchsian differential equation with four regular singularities by making three of these singularities coalesce at infinity.[15] This equation is known as the biconfluent Heun equation [42]. With the change of variables

$$\chi = e^{-\frac{1}{2}u^2}\,Y(u)\,, \qquad u = \sqrt{P}\,s\,, \tag{B.3}$$

equation (B.1) is mapped into a standard form

$$uY'' + (1+\alpha-\beta u - 2u^2)\,Y' + \Big((\gamma - 2 - \alpha)\,u - \frac{1}{2}(\delta + \beta(1+\alpha))\Big)\,Y = 0\,, \tag{B.4}$$

with $(\alpha,\beta,\gamma,\delta) = \big(\alpha, 0, 0, 4P^{3/2}\big)$. The solution regular at $s = 0$ is commonly denoted as $N(\alpha,\beta,\gamma,\delta;u)$. For $\beta = \delta = 0$, it reduces to a hypergeometric function; equation (B.1) is a different (but also very special) case.

Despite considerable effort, see e.g. [42] and references therein, the Heun equation is still far less understood than the hypergeometric equation, which is the analogous fluctuation equation in the previously studied flows. For the purposes of computing correlation

[15] For comparison, the confluent hypergeometric equation has s-rank $\{1;2\}$.

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functions, there is even an additional technical complication here that comes from the fact that the relevant expansion (B.2) (i.e. the UV boundary of the flow) is around a generic regular point ($s = 1$), rather than around a singular point as in the higher-dimensional examples [11, 28]. Expanding around a singular point considerably simplifies the resulting expressions; the coefficient analogous to Ψ in (B.2) for the case of the hypergeometric equation is usually denoted as ψ and is simply expressed in terms of Γ-functions. In all, it is a tall order to solve (B.1), but we find that we can extract the important information analytically, and use numerics to check.

Following [42], we denote by $H^+(\alpha, \beta, \gamma, \delta; u)$ the unique solution of (B.4) that is regular as $s \to \infty$. The relevant coefficient in (B.2) is then given by

$$\Psi_\alpha(P) = -P + \frac{\partial}{\partial s} \log H^+ \left(\alpha, 0, 0, 4P^{3/2}; \sqrt{P}\, s\right)\Big|_{s=1} . \tag{B.5}$$

Analytic expressions for $H^+(\alpha, \beta, \gamma, \delta; u)$ may be constructed along the lines of [43] as an infinite chain of sums over Pochhammer symbols $(x)_n = \Gamma(x + n)/\Gamma(x)$.[16] Rather than constructing these series, we will derive analytical results only for the IR asymptotics in P, and use the numerical solution of (B.5) for other purposes.

To this end, we consider the following change of variables

$$\chi = v^{-\alpha/4}\, Z(v)\,, \qquad v = \frac{1}{2} P\, s^2\,, \tag{B.6}$$

which transforms (B.1) into

$$\Delta_{\alpha/4} Z \equiv v^2 Z'' + v Z' - \left(v^2 + \frac{\alpha^2}{16}\right) Z = P^{3/2}\, \frac{\sqrt{v}}{\sqrt{2}}\, Z\,. \tag{B.7}$$

where prime now denotes a derivative with respect to v. This equation may thus be resolved into a series of inhomogeneous Bessel equations

$$Z = \sum_{r=0}^{\infty} P^{3r/2}\, 2^{-r/2}\, Z_r\,, \qquad \Delta_{\alpha/4}\, Z_r = v^{1/2}\, Z_{r-1}\,. \tag{B.8}$$

which can successively be integrated in terms of Lommel functions, demanding regularity in the interior $v = \infty$. Note that this expansion is, in particular, compatible with the IR (small P) asymptotics. For instance, for $\alpha = -2$, we explicitly find the first terms in this series as

$$Z_0 = \sqrt{\frac{2}{\pi}}\, K_{1/2}(v) = v^{-1/2} e^{-v}\,, \qquad Z_1 = \sqrt{2\pi}\, v^{-1/2} e^{v}\, \mathrm{erfc}(\sqrt{2v})\,, \tag{B.9}$$

with the complementary error function $\mathrm{erfc}(v)$. After some computation, this gives rise to the small P asymptotics

$$\Psi_{-2} = -\frac{\alpha}{2} + P\, \frac{\partial}{\partial v} \log Z\Big|_{v=P/2} = -P - P^2 + 2\sqrt{\pi}\, P^{5/2} - 4P^3 + \cdots\,. \tag{B.10}$$

[16]Unfortunately, the earlier results of [44] which seemingly give $N(\alpha, 0, 0, \delta)$ as a series in hypergeometric functions are incorrect; the double sum in eq. (2.8) in that paper does not, in fact, factor into (2.10); a relation which is recursively used in the construction.

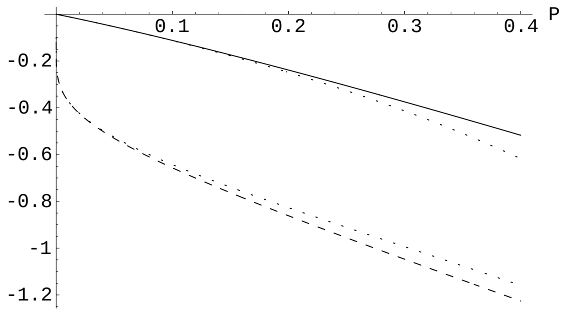

Figure 3: Small P (IR) asymptotics of Ψ_{-2} (straight), and Ψ_0 (dashed). The dotted lines correspond to the first terms of the exact asymptotics (B.10), (B.11)

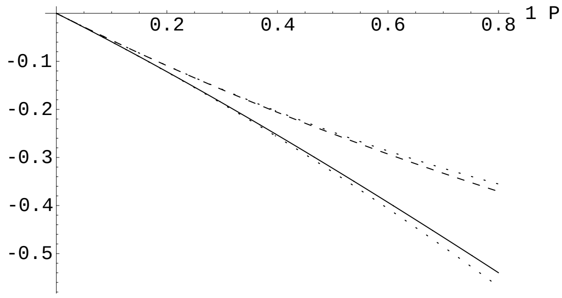

Figure 4: Large P (UV) asymptotics of $1/\Psi_{-2}$ (straight), and $1/\Psi_0$ (dashed). The dotted lines correspond to the first terms of the exact asymptotics (B.12). The horizontal axis is $1/P$.

For $\alpha = 0$, one finds similarly the lowest term $Z_0 = K_0(v)$, and thus the small p asymptotics

$$\Psi_0 = \frac{2}{C + \log(P/4)} + \cdots \,, \tag{B.11}$$

with Euler's constant C. As we have shown in the main text, the two-point correlation functions of the inert scalars in the **4** are proportional to the inverse of Ψ_0; the log-term then describes the standard behavior of a dimension $\Delta = 1$ operator in the IR, cf. table 1. An analysis of the large P asymptotics of the Ψ_α gives

$$\Psi_{-2} = -\sqrt{3}\,P + \frac{1}{2} + \cdots \,, \qquad \Psi_0 = -\sqrt{3}\,P - \frac{1}{2} + \cdots \,. \tag{B.12}$$

To obtain a numerical expression for the important ratio Ψ_α, we can use the following simple prescription. Consider the differential equation (B.4) and numerically compute two of its solutions with two different sets of initial conditions given at $u = \sqrt{P}$ (the regular point $s = 1$) as

$$\begin{aligned} Y_1(\sqrt{P}) &= 0\,, \qquad Y_1'(\sqrt{P}) = -\sqrt{P}\,,\\ Y_2(\sqrt{P}) &= 1\,, \qquad Y_2'(\sqrt{P}) = 0\,. \end{aligned} \tag{B.13}$$

Since there exists a unique solution regular as $u \to \infty$ (called $H^+(u)$ above), the ratio Y_2/Y_1 will tend to a constant in this limit, and this constant may be determined numerically. From (B.5), we then have the relevant coefficient Ψ_α as

$$\Psi_\alpha = -P + \lim_{u\to\infty} \frac{Y_2}{Y_1}\,. \tag{B.14}$$

The result for $\alpha = -2, 0$ is plotted in figures 3 and 4 together with the first few terms of the exact asymptotics, given in (B.10)–(B.12). Agreement in these regions is already quite good for including just a few terms.

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